High order quadratures for the boundary integrals governing axisymmetric interface motion

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Boundary integral methods have been used widely for the simulation of interfacial flows. They are Lagrangian free surface representations that reduce the problem to one defined solely on the fluid interface. In particular, in either inviscid flow or in Stokes flow, the fluid velocity is given by integrals whose domain is the interface. Boundary integral methods consist of approximating these integrals and evolving the interface with the approximate velocity.

In planar and axisymmetric geometries the surfaces are represented by a curve in the plane. Let $(x(\alpha,t),y(\alpha,t)), \alpha\in[a,b]$ be a parametrization of this curve. Then the interface velocity at $\alpha=\alpha_j$ is given by

$$\begin{aligned}\frac{dx}{dt}(\alpha_j,t) &= \int_a^b G^u(\alpha,\alpha_j,t)d\alpha \\ \frac{dy}{dt}(\alpha_j,t) &= \int_a^b G^v(\alpha,\alpha_j,t)d\alpha\end{aligned} \tag{1}$$

for some functions G^u, G^v that are generally singular at $\alpha=\alpha_j$ and whose precise form depends on the fluid under consideration, the properties at the interface, and the geometry. The integrals on the right hand side of (1) are usually approximated using a generalized trapezoid rule [1] based on a uniform discretization of the curve. For planar smooth interfaces, the integrals can be approximated with spectral accuracy (see eg, [2-5]) However, for axisymmetric surfaces that cross the axis of symmetry, a complication occurs that is not present in the planar case. The integrands $G^{u,v}$ have large variations which make it difficult to evaluate the velocity accurately [3,6,7]. For definiteness, suppose the parameter values $\alpha=a,b$ denote those points on the interface that lie on the axis of symmetry. In [8], we showed that as $\alpha,\alpha_j\to a,b$ the derivatives of G at the endpoints $\alpha=a,b$ grow unboundedly in such a way that high order quadratures have maximum errors of, at best, first order accuracy in the case of inviscid flow, and second order accuracy in the case of Stokes flow [8,9,10]. That is, these quadrature rules are pointwise of high order accuracy, but not uniformly for all $\alpha_j\in[a,b]$. As a result, large errors introduced near the axis of symmetry contaminate the overall results.

We note that the singularity of the integrands at the axis of symmetry is introduced solely by the singular behaviour of the axisymmetric coordinate system on the axis.

We resolve the difficulty described above using the approach presented in [8] and [10]. In those papers we find, using asymptotic analysis, that the integrand can be approximated near the axis by functions which capture the singular behaviour and are, up to constant multipliers, time-independent. For example, for $\alpha,\alpha_j\approx a$,

$$G(\alpha,\alpha_j,t)\approx B(\alpha,\alpha_j,t)=b_0(t)B_0\Big(\frac{\alpha}{\alpha_j}\Big)+\alpha_j^2\sum_{k=1}^{3}b_k(t)B_k\Big(\frac{\alpha}{\alpha_j}\Big)+O\big((\alpha-a)^4,(\alpha_j-a)^4\big)$$

for inviscid flow [8], and

$$G(\alpha,\alpha_j,t)\approx B(\alpha,\alpha_j,t)=\alpha_j b_1(t)B_0\Big(\frac{\alpha}{\alpha_j}\Big)+\alpha_j^3\sum_{k=2}^{6}b_k(t)B_k\Big(\frac{\alpha}{\alpha_j}\Big)+O\big((\alpha-a)^5,(\alpha_j-a)^5\big)$$

for Stokes flow [10], where all coefficients $b_k(t)$ and functions $B_k(\eta)$ are obtained by the asymptotic expansion. Using these approximations, we can compute time-independent corrections that eliminate the large errors introduced near the axis and enable uniformly accurate simulations, at no additional cost per time-step.

The resulting uniform quadrature rules in inviscid flow have been applied to resolve singularity formation in axisymmetric vortex sheet flow [9], capillary pinchoff in a soapfilm model [11], and Hele-Shaw flow with suction [12]. In this paper we apply the uniform approximations in Stokes flow to resolve (1) bubble pinchoff and (2) steady state solutions, using efficient and accurate computations.

References

[1] A. Sidi & M. Israeli, *Quadrature methods for periodic singular and weakly singular Fredholm integral equations*, J. Sci. Comp, **3** (1988), 201–231.

[2] A. I. van de Vooren, *A numerical investigation of the rolling-up of vortex sheets*, Proc. R. Soc. Lond. A, **373** (1980), 67-91.

[3] G. R. Baker, D. I. Meiron, & S. A. Orszag, *Boundary integral methods for axisymmetric and three-dimensional Rayleigh-Taylor instability problems*, Physica 12D (1984), 19-31.

[4] M. J. Shelley, *A study of singularity formation in vortex sheet motion by a spectrally accurate vortex method*, J. Fluid Mech., **244** (1992), 493-526.

[5] T. Y. Hou, J. S. Lowengrub & M. J. Shelley, *Removing the Stiffness from Interfacial Flows with Surface Tension*, J. Comp. Phys. **114** (1994), 312–338.

[6] B. de Bernadinis, J. M. R. Graham, & K. H. Parker, *Oscillatory flow around disks and through orifices*, J. Fluid Mech., **102** (1981), 279-299.

[7] D. A. Pugh, *Development of vortex sheets in Boussinesq flows - formation of singularities*, Ph. D. Thesis, Imperial College of Science and Technology, London (1989).

[8] M. Nitsche, *Axisymmetric vortex sheet motion: accurate evaluation of the principal value integral*, SIAM J. Sci. Comput. **21**, No 3 (1999), 1066–1084.

[9] Nitsche, M, *Singularity formation in a cylindrical and a spherical vortex sheet*, J. Comp. Phys. **173** (2001), pp. 208–230.

[10] Nitsche, M. & Steen, P. H. *Numerical simulations of inviscid capillary pinchoff*, J. Comp. Phys. **200** (2004), 299–324.

[11] H. D. Ceniceros, A. L. Karniala & M. Nitsche, *High order quadratures for the evaluation of interfacial velocities in axi-symmetric Stokes flows*, submitted.

[12] H. D. Ceniceros, T. Y. Hou and H. Si, *Numerical study of Hele-Shaw flow with suction*, Phys. Fluids **11**, No. 9 (1999), 2471-2486