Polynomials and Soliton Equations

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There has been considerable interest in completely integrable partial differential equations (PDEs) solvable by inverse scattering, the *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal, and Miura [18] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

During the past thirty years or so there have been several studies of rational solutions for the soliton equations, applications of which include the description of explode-decay waves [21] and vortex solutions of the complex sine-Gordon equation [6, 23]. Airault, McKean, and Moser [4] studied the motion of the poles of rational solutions of the KdV equation (1) and the Boussinesq equation

$$u_{tt} + (u^2)_{xx} \pm u_{xxxx} = 0, \tag{2}$$

and related the motion to an integrable many-body problem, the Calogero-Moser system with constraints; see also [3]. Ablowitz and Satsuma [1] derived rational solutions of the KdV equation (1) and the Boussinesq equation (2) by finding a long-wave limit of the known N-soliton solutions of these equations.

The *Painlevé equations* are six nonlinear ordinary differential equations (ODEs) discovered by Painlevé, Gambier and their colleagues around the beginning of the 20th century. Their solutions define new transcendental functions as they are not expressible in terms of previously known functions such as elementary and elliptic functions or in terms of solutions of linear ODEs and can be thought of as nonlinear analogues of the classical special functions [12, 16, 19, 24]. Ablowitz and Segur [2] demonstrated a close relationship between the soliton and Painlevé equations. The second Painlevé equation ($\mathrm{P_{II}}$)

$$w'' = 2w^3 + zw + \alpha, \tag{3}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α is an arbitrary constant, arises as a scaling reduction of the KdV equation (1) and the fourth Painlevé equation ($\mathrm{P_{IV}}$)

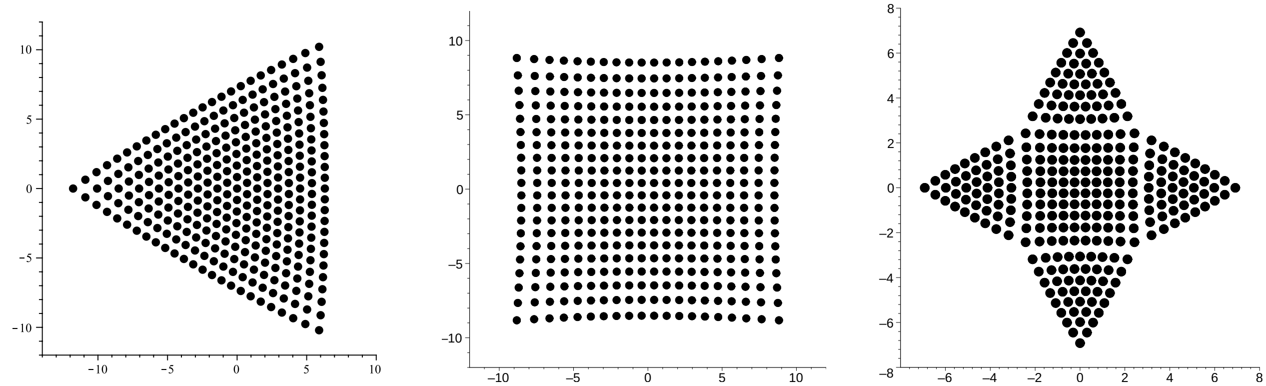
$$ww'' = \tfrac{1}{2}\,(w')^2 + \tfrac{3}{2}w^4 + 4zw^3 + 2(z^2-\alpha)w^2 + \beta, \tag{4}$$

where α and β are arbitrary constants, arises as scaling reductions of the Boussinesq equation (2) and the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0. \tag{5}$$

Vorob'ev and Yablonskii [25] expressed the rational solutions of $\mathrm{P_{II}}$ in terms of polynomials, now known as the *Yablonskii–Vorob'ev polynomials*. Noumi and Yamada [22] expressed rational solutions of $\mathrm{P_{IV}}$ in terms two sets of polynomials, the *generalized Hermite polynomials* and the *generalized Okamoto polynomials*. Clarkson and Mansfield [15] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these roots have a very regular, triangular structure (see Figure 1a). The structure of the (complex) roots of the polynomials associated with rational solutions of $\mathrm{P_{IV}}$ is studied in [9], which either have a rectangular structure and or are a combination of rectangular and triangular structures (see Figures 1b,1c). Polynomials associated with rational solutions of the third and fifth Painlevé equations are discussed in [10]. The Yablonskii–Vorob'ev polynomials arise in string theory [20], and the generalized Hermite polynomials arise in random matrix theory [7, 17] and the theory of orthogonal polynomials [8].

These polynomials associated with rational solutions of the Painlevé equations are related to polynomials associated with rational solutions of soliton equations. The Yablonskii–Vorob'ev polynomials are special cases of the *Adler-Moser polynomials* [3, 4], which describe rational solutions of the KdV equation (1). The generalized Hermite and generalized polynomials respectively are special cases of the polynomials which describe rational solutions of the



a b c

Figure 1: Roots of polynomials associated with rational solutions of P_{II} (a) and P_{IV} (b,c)

NLS equation (5) and Boussinesq equation (2), which have a different structure to the Adler-Moser polynomials (see [11, 13, 14]).

Aref [5] discusses this connection between point vortex dynamics and polynomials with roots at the positions of the vortices. For stationary vortex configurations the following results have been established: (i), n identical vortices on a line are in equilibrium if and only if they are situated at the roots of the classical nth Hermite polynomial; (ii), n identical vortices on a circle are in equilibrium if and only if they are situated at the vertices of a regular n-polygon; (iii), $\frac{1}{2}n(n+1)$ positive and $\frac{1}{2}n(n-1)$ negative vortices are in equilibrium if and only if they are situated at the roots of the Adler-Moser polynomials arise in the description of stationary vortex patterns. Further Aref [5] remarks that the relationship between vortex dynamics and the KdV equation (1) is "quite unexpected and very beautiful".

An interesting question is whether there is a the relationship between vortex dynamics and polynomials associated with rational solutions of other soliton equations, such as the the NLS and Boussinesq equations. Since the structure of rational solutions of the NLS and Boussinesq equations is different to those of the KdV equation then possibly other vortex dynamics can be expressed in terms of the associated polynomials.

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