About finite area wakes past bluff bodies and growing vortex patches

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Abstract

Vorticity, ω, holds a constant value on $\psi = const$ streamlines of inviscid 2D flows, that is $\omega = \omega(\psi)$. The equation governing the flow is then the non linear Poisson equation:

$$\nabla^2\psi = -\omega(\psi) \tag{1}$$

The value of the vorticity on closed streamlines is not defined by the the far field boundary conditions; as a consequence, for finite area wakes, this equation provides multiple solutions for separated flows past bluff bodies. The multiplicity of solutions is relative to the different distributions of vorticity $\omega(\psi)$ which can be assumed for the region with closed streamlines. By assuming as a simple model for the vorticity distribution a two level piecewise-constant distribution, the wake consists of an inner core patch, with $\omega \neq 0$ and a surrounding $\omega = 0$ potential flow. We argue that these wakes form a three parameter family, with the parameters being the area A of the core, the value of the vorticity ω or, equivalently, of the circulation Γ and the jump ΔB of the Bernoulli constant across the vortex sheet that separates the core from the potential flow.

The assumption is verified for $A = 0$ solutions, which are pertinent to standing point vortices. In these cases ΔB is not defined. In [1] it has been shown that past any protruding body from a wall, there is a locus of standing point vortices. Such locus is the generalization of the Föppl curve pertinent to semicircular obstacles. Thus, each standing point vortex is relevant to a finite area wake characterized by $A = 0$ and by a value of the circulation Γ.

We base our work on the hypothesis that such point vortex solutions can be considered as seeds from which the above three parameter family can grow, that is, solutions with different values of A and ΔB and the same Γ can be obtained by continuation from each standing vortex.

Such conjecture has a physical interest, in fact, if true, it relates the point vortex solutions, which are easily detected, to the Batchelor [2] flow solution, which possesses a strong physical meaning. Actually, Batchelor [2] has shown that the limit solution of the viscous flow for the Reynolds number going to infinity is characterized by $\omega(\psi) = const$ in the region with closed streamlines, that is, that the finite area wake reduces to a vortex patch. Moreover, he has shown that the value of the vorticity in this region is not arbitrary and can be found by taking into account the boundary layer.

In [1], it is shown that when an obstacle presents a sharp edge, then there is a finite or null number of standing point vortices that satisfy the Kutta condition. It is conjectured that if there is not a standing point vortex that satisfies the Kutta condition, as, for instance, in the case of a normal flat plate, then the associated family of growing patches, including the Batchelor-like vortex patch, does not exist either. This conjecture implies that this kind of obstacles do not admit a finite area wake at high Reynolds number.

This assertion contradicts several results present in the literature. In the flow past a normal plate, Turfus [3] numerically detected a finite area vortex patch by explicitly computing the shape of the recirculation region. The existence of a closed wake for the inviscid flow past normal plates was also discussed by Turfus & Castro [4], who demonstrated that a cyclic boundary layer is compatible with a the finite area solution determined previously by Turfus. More recently Castro [5] obtained computational results suggesting the possible existence of a second branch of the graph representing the wake size versus the flow Reynolds number, which would extrapolate to a finite area vortex in the limit of an inviscid flow.

We want to give some numerical consistency to the above conjecture on non existence of a finite area wake if there is not a standing vortex. Our opinion is that the controversial results found in literature are due to poor convergence of numerical results. To this purpose, taking inspiration from [6], we developed a mixed analytical numerical methods to attain high accuracy.

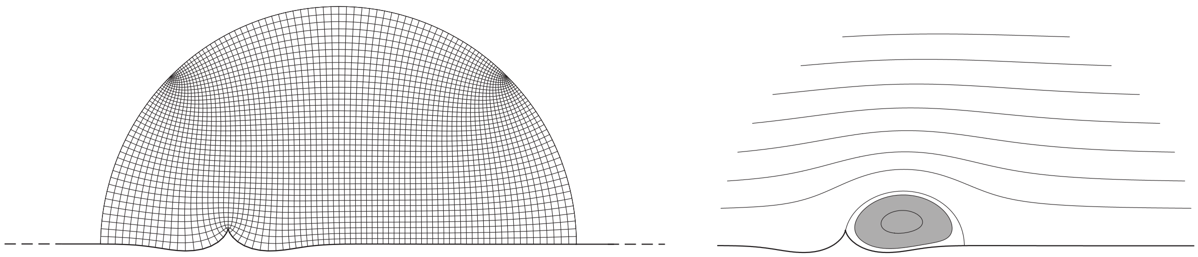

Figure 1: Two computational domains - Streamlines and vortex patch in equilibrium

Our numerical approach is based on a Steklov-Poincaré iteration. The flow field, which extends to infinity, is decomposed into two domains, one external to the wake, governed by $\nabla^2\psi = 0$ and the other one, internal, which contains the wake. Inside, the governing equation is

$$\nabla^2\psi = -H(\psi - \psi_o)\omega, \tag{2}$$

where H is the Heaviside function and ψ_o is the value of the stream function on the patch contour. ψ_0 is, thus, a parameter which is equivalent to the area A. By assuming ΔB=0, ω is found as part of the solution by enforcing the Kutta condition.

The internal region is shown on the left side of fig.1). According to a Steklov-Poincaré iteration, the external flow is solved analytically, by imposing as boundary conditions the far field condition, $\psi = const$ at the solid wall and, on the common boundary, the Neumann condition $\partial\psi/\partial n$, deducted by the inner solution. The internal flow is solved numerically by enforcing the Dirichlet boundary condition, which on the solid wall prescribes a constant value to ψ and prescribes on the common boundary the value deduced from the external solution. The grid, coarser than the one used in the computations, is shown in the figure. It has been obtained by conformally mapping a rectangle onto a circle. On the right side of fig.1), a typical solution of the wake containing a patch is shown.

These results seem to support that the perturbation equations to eq.(2) with respect to ω and ψ_0 always admit solution:

$$\nabla^2\left(\frac{\partial\psi}{\partial\psi_0}\right) = -\omega\left(1 - \frac{\partial\psi}{\partial\psi_0}\right)\delta\left(\psi_0 - \psi\right)$$

$$\nabla^2\left(\frac{\partial\psi}{\partial\omega}\right) = -H\left(\psi_0 - \psi\right) + \omega\frac{\partial\psi}{\partial\omega}\delta\left(\psi_0 - \psi\right),$$

as it is discussed in the full paper.

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