Short-wave Stability of a Helical Vortex Tube: the Effect of Torsion on the Curvature Instability

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Abstract

In this paper we study the short-wave stability of a helical vortex tube. The base flow field inside the helical vortex tube is obtained by perturbation expansion assuming that ε, the ratio of the core to curvature radius of the helical tube, be small. Short-wavelength stability analysis is applied with modification required for the present configuration of helical vortex tube. It is shown that the helical vortex tube suffers from curvature instability found for vortex rings; the effect of torsion on the stability appears at the second order of ε.

Helical tip vortices commonly appear in the wakes of helicopter rotors, wind turbines and other devices which have rotating wings. Since the helical tip vortices have a vital influence on those devices as exemplified by the aerodynamic performance, noise emission and cavitation, it is undoubtedly important to study their basic properties. We can legitimately model them by a helical vortex tube which extends infinitely.

There are many works on the motion of a helical vortex filament of which thickness is assumed to be very small. For example, Widnall[1] considered a filament with non-zero thickness to obtain the self-induced flow field; but she studied the stability of the helical vortex filament by analyzing the sinusoidal motion of the centerline of the filament, which implies that dynamical motion inside the filament is neglected. In this sense the stability of the helical vortex filament studied so far is the *long-wave* stability. As we have shown in the case of vortex ring[2, 3], however, curvature effect of a helical tube could cause the *short-wave* instability. A helical vortex tube would be the simplest vortical structure featured not only by curvature but also by torsion. The effect of torsion on the stability has been left unaddressed, though vortical structures in real flow are more or less curved and twisted by various effects.

We consider a helical vortex tube whose centerline is a helix of constant curvature and torsion in an inviscid and incompressible fluid (Figure 1(a)). We assume that the tube have a circular core of finite thickness; when we consider the helical vortex tube in an unbounded domain, this assumption is valid up to $O(\varepsilon)$ but is not the case at $O(\varepsilon^2)$[4]. Here we impose this assumption in order to focus on the pure effects of torsion and minimize other effects like self-induced strain.

There exist tubes which translate and rotate due to the self-induced velocity but do not change the shapes. In other words, steady solutions can be found for the Euler equation in a rotating frame. We use a helical coordinate system along the tube (r, θ, s) (Figure 1(a)). We solve the equation by perturbation expansion. Substituting the flow field expanded as $\boldsymbol{u} = \boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \varepsilon^2 \boldsymbol{u}_2 + \cdots$ into the Euler equation expressed in the helical coordinate system, we obtain equations at each order of ε. The leading-order solution is set to be a solid body rotation as in the case of the vortex ring. Then the solutions read

$$\boldsymbol{u} = \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \frac{5}{8}\left(1-r^2\right)\sin\varphi \\ \left(\frac{5}{8}-\frac{7}{8}r^2\right)\cos\varphi \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \left(\frac{r}{8}-\frac{r^3}{8}\right)\sin 2\varphi \\ \left(\frac{r}{8}-\frac{r^3}{16}\right)\cos 2\varphi \\ \left[\left(\frac{5}{8}\alpha+\frac{\beta}{\sqrt{1+\alpha^2}}\right)r-\frac{3}{8}\alpha r^3\right]\cos\varphi \end{pmatrix} + O(\varepsilon^3), \tag{1}$$

where α and β are non-dimensionalized torsion and rotation rate. At $O(\varepsilon)$ the flow field is the same with that of a vortex ring except that the angle θ is replaced with $\theta - \varepsilon\alpha s$; the first-order dipole flow is twisted due to torsion. At $O(\varepsilon^2)$ torsion gives rise to the velocity component parallel to the centerline. Here we confine ourselves to the major effect of the torsion, the twisting of the

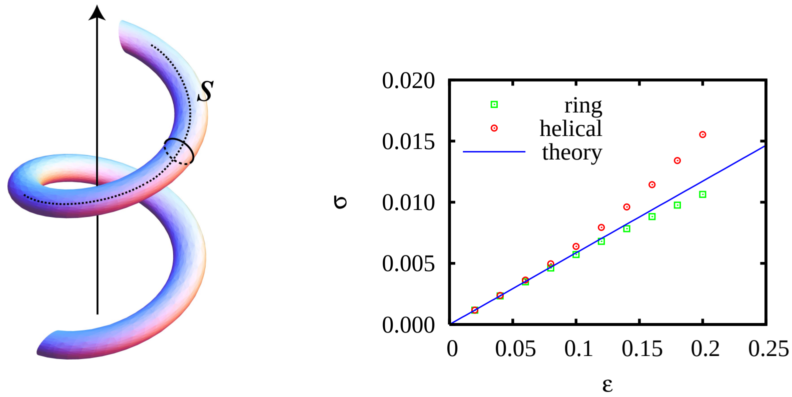

Fig. 1: (a) Helical vortex tube. (b) Growth rates obtained by short-wavelength stability analysis.

first-order dipole flow, on the short-wavelength stability. An immediate consequence is that the curvature instability is present for the helical vortex tube since the equation and the base flow are the same with the case of a vortex ring up to $O(\varepsilon)$.

In order to study the short-wavelength stability of a helical vortex tube we invoke the geometric optics method. It gives asymptotic growth rates in the short wave limit. The task is to solve the following system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}), \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\mathcal{L}^T\boldsymbol{k}, \quad \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \left(\frac{2\boldsymbol{k}\boldsymbol{k}^T}{|\boldsymbol{k}|^2} - I\right)\mathcal{L}\boldsymbol{a}, \tag{2}$$

where $\mathcal{L}_{ij} = \frac{\partial u_i}{\partial x_j}$, and to see the evolution of the amplitude $\boldsymbol{a}$. Although the equation for $\boldsymbol{a}$ has three components, the incompressibility condition imposes $\boldsymbol{k}\cdot\boldsymbol{a} = 0$ which implies that the number of degrees of freedom is two. In the case of a helical vortex tube it is convenient to choose the independent variables in the same way as the case of two-dimensional base flow[5]: $p = (k/k_\perp)\boldsymbol{k}_\perp\cdot\boldsymbol{a}_\perp$, $q = (k/k_\perp)(\boldsymbol{k}_\perp\times\boldsymbol{a}_\perp)\cdot\boldsymbol{e}_s$, where the subscript $\perp$ denotes the projection to $r\theta$-plane. Then the equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p \\ q \end{pmatrix} = A(t)\begin{pmatrix} p \\ q \end{pmatrix}, \quad A(t) = A_0 + \varepsilon A_1(t) + \varepsilon^2 A_2(t) + \cdots. \tag{3}$$

The leading-order matrix A_0 coincides with the expression in Bayly *et al.*[5]; the first-order matrix $A_1(t)$ is same with the case of a vortex ring[3]; the second-order matrix $A_2(t)$ includes terms proportional to torsion and peculiar to the helical vortex tube.

Here we show a numerical result obtained by solving the above system with $\boldsymbol{u}$ being the base flow obtained up to $O(\varepsilon^2)$ by perturbation expansion. Figure 1(b) shows an example of the growth rate of the curvature instability for a vortex ring and a helical vortex tube. Notably, the torsion enhances the instability in this case. More detailed numerical results and analytical results will be presented in the symposium.

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