VORTEX DYNAMICS IN A TWO DIMENSIONAL DOMAIN WITH HOLES

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Abstract. The purpose of this article is to describe a new formulation of vortex dynamics in domains with holes, where the evolution of the potential part of the flow is determined explicitly using certain harmonic moments of vorticity.

Let Ω be a bounded connected domain in the plane whose boundary $\partial\Omega$ is the disjoint union of a finite number of smooth curves Γ_i, $i = 0, 1, \ldots, k$. We assume Ω is the bounded region with boundary Γ_0 with the bounded regions bounded by Γ_i, $i = 1, \ldots, k$ removed.

A vector field in Ω is called *potential* if it is both divergence and curl free. We denote the vector space of potential vector fields, tangent to the boundary of Ω by $\mathcal{H}$. It is a consequence of Hodge's Theorem that the vector space $\mathcal{H}$ is k-dimensional, and that there is a unique family of potential vector fields $\{X_1, X_2, \ldots, X_k\}$ satisfying the period conditions

$$\oint_{\Gamma_j} X_i \cdot ds = \delta_{ij}, \text{ with } i = 1, \ldots, k \text{ and } j = 0, 1, \ldots, k, \tag{1}$$

which are a basis for $\mathcal{H}$.

For each $i = 1, \ldots, k$, we introduce the function ϕ_i as the unique solution to the Dirichlet problem

$$\begin{cases} \Delta\phi_i = 0 \\ \phi_i = 0 \text{ on } \Gamma_j, j \neq i \\ \phi_i = 1 \text{ on } \Gamma_i \end{cases} . \tag{2}$$

An alternative basis for $\mathcal{H}$ is given by $\{(-\partial_{x_2}\phi_i, \partial_{x_1}\phi_i)\}_{i=1}^k$.

Given $\omega \in C^\infty(\overline{\Omega})$, let ψ be the unique solution of the Poisson problem $\Delta\psi = \omega$ in Ω, with $\psi = 0$ on $\partial\Omega$. We define the Biot-Savart operator $K_\Omega[\omega] \equiv (-\psi_{x_2}, \psi_{x_1})$.

Fix $u \in L^2(\Omega)$, divergence-free and tangent to the boundary and let $\omega \equiv$ curl u. The Hodge-Kodaïra Decomposition Theorem implies that u can be decomposed in an unique manner in the form $u = \nabla^\perp\psi + H$, with ψ smooth, vanishing on $\partial\Omega$ and $H \in H(\Omega)$. This deconposition is orthogonal with respect to the L^2 inner product. Using this decomposition, u can be written in an unique way as:

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$$u = K_\Omega[\omega] + \sum_{j=1}^{k} \alpha_j X^j, \tag{3}$$

for some $\alpha_j \in \mathbb{R}$.

In [1], the author proved the following result

Proposition 1. *With the notation above, for each $i = 1, \ldots, k$ we have*

$$\alpha_i = \int_\Omega \phi_i \omega dx + \oint_{\Gamma_i} u \cdot ds.$$

Using this result and Kelvin's Circulation Theorem, we can write the following initial value problem, with only ω as unknown.

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0 \text{ in } \Omega \times (0, T) \\ u = K_\Omega[\omega] + \sum_{j=1}^{k} \alpha_j(t) X_j \text{ in } \Omega \times \{t\}, \text{ for } 0 \leq t < T \\ \alpha_j(t) = \int_\Omega \phi_j \omega dx + \alpha_{j,0} - \int_\Omega \phi_j \omega_0 dx \text{ in } [0, T) \\ \omega(x, 0) = \omega_0(x) \text{ and } \alpha_j(0) = \alpha_{j,0} \text{ in } \Omega \times \{t = 0\}, \end{cases} \tag{4}$$

The initial data for problem (4) is the initial vorticity ω_0 and the initial coefficients for the harmonic part of velocity, $\alpha_{j,0}$. Clearly, assigning this data is equivalent to choosing and initial velocity u_0. System (4) is the vortex dynamics formulation of the ideal flow equations on Ω.

A natural alternative to using Proposition 1 to write a vortex dynamics equation in a two dimensional domain with holes is to obtain an equation for the evolution of α_j by multiplying the momentum equation by X_i and integrating in Ω. The resulting equation is complicated, difficult to evaluate numerically and also difficult to express in terms of point vortex dynamics. System (4) was used by the author to study small obstacle asymptotics for 2D ideal flows in [1].

To obtain the point vortex system corresponding to (4), for each $x \in \Omega$, let $U = U(x, y) \equiv K_\Omega[\delta(\cdot - x)](y)$, the Biot-Savart velocity generated at y by a unit mass point vortex placed at x. Let $Q_i = Q_i(t)$ denote the trajectory of a point vortex of mass m_i, for $i = 1, \ldots, n$. The point vortex equations are

$$\dot{Q}_i = \sum_{j \neq i} m_j U(Q_j, Q_i) + \sum_{j=1}^{k} \left(\sum_{k=1}^{n} m_k (\phi_j(Q_k) - \phi_j(Q_k(0)) + \alpha_{j,0} \right) X_j(Q_i).$$

[1] M. C. Lopes Filho, *Two dimensional vortex dynamics in a domain with holes and the small obstacle limit*, SIAM J. Math. Anal. **39** (2007), 422-436.

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