Finite area vortex motion on a sphere with impenetrable boundaries

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Abstract

Techniques are presented for computing the motion of a finite area patch of vorticity on a sphere with impenetrable boundaries based on conformal mapping and the numerical method of contour dynamics. Several examples of impenetrable boundaries are considered including a spherical cap, longitudinal wedge, half-longitudinal wedge and a thin barrier. Finite area patch motion is compared to exact point vortex trajectories for cases where the patch remains close to circular. Good agreement is found between the point vortex trajectories and the centroid motion of finite area patches. More exotic motion of the finite area patches, particularly in the thin barrier case, is then examined. In this case, when a background flow parallel to the barrier is imposed and the vortex path pushed close to one of the barrier edges, vortex shedding and even splitting can occur and in certain cases this can give rise to a quasi-steady trapped vortex.

1 Introduction

The study of vortex motion in bounded domains with impenetrable boundaries is an important problem in vortex dynamics due to its relevance in modeling geophysical flows, especially oceanographic flows. Oceanic eddies frequently interact with topography such as ridges and coastlines. For localized large-scale vortex structures, the curvature and rotation of the Earth can play a significant role in the evolution of the system. Johnson & McDonald [5] consider the planar problem of a point vortex near a gap in a wall with and without ambient flows for which exact solutions of point vortex trajectories are obtained. They then present a method for computing the motion of finite area patches of constant vorticity near such a gap for which conformal mapping and the numerical method of contour dynamics are utilized. Dritschel [3] extends the method of contour dynamics to the surface of a sphere. Dritschel & Polvani [4] and Polvani & Dritschel [6] then utilize this method to model waves and vortices on a sphere.

Kidambi & Netwon [8] and Crowdy [1] consider the motion of a point vortex on the surface of a sphere with impenetrable boundaries. In particular, [8] utilize the method of images to derive exact point vortex trajectories within a spherical cap, longitudinal wedge, half-longitudinal wedge, channel and rectangle. Due to limitations of the method of images, solutions for the wedge, half-wedge and rectangle are restricted to bounded geometries between the angles of $\phi = 0$ and $\phi = \pi/m$ where m is a positive integer. Crowdy [1] presents a formulation based on a generalization of the Kirchhoff-Routh path function (see Saffman [7]) to the surface of a sphere, abandoning the need for special symmetries and also present the point vortex problem considered in Ref. [5] on the surface of a sphere.

The present work considers the motion of finite area patches of constant vorticity on the non-rotating unit sphere with impenetrable boundaries. Techniques described in Ref. [5] are extended to the surface of the sphere and used to solve for the motion of a vortex patch about an ellipse of semimajor axis a and semiminor axis b in the stereographic plane, and within a longitudinal wedge and half-longitudinal wedge bounded between zero and an angle ϕ (where $0 < \phi < 2\pi$). For the case when the semiminor axis of the ellipse is equal to zero the motion of the patch with compatible background flows is also considered.

2 Problem formulation

The system consists of a single finite area vortex patch of circulation Γ on the surface of the unit sphere with an impenetrable boundary. The domain $\mathcal{D}$ on the spherical surface is assumed to be simply connected and on the boundary $\partial\mathcal{D}$ the no-normal-flow boundary condition, $\mathbf{u}\cdot\mathbf{n} = 0$, applies. Let $\tilde{\psi}$ be the the streamfunction owing to the vortex patch in the domain $\mathcal{D}$ such that

$$\tilde{u}_\theta = -\frac{1}{\sin\theta}\frac{\partial\tilde{\psi}}{\partial\phi} \quad , \quad \tilde{u}_\phi = \frac{\partial\tilde{\psi}}{\partial\theta}, \tag{1}$$

where $\phi\epsilon[0, 2\pi]$ is the azimuthal angle and $\theta\epsilon[0, \pi]$ is the latitudinal angle. From the equations of contour dynamics on a sphere (Ref. [3]) the velocities owing to $\tilde{\psi}$ at $\partial\mathcal{D}$ can be found. The task then becomes one of finding an

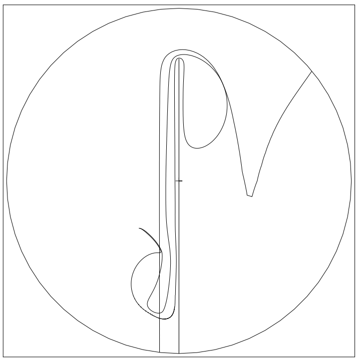

Figure 1: Vortex patch with $\Gamma \approx -3.8$ and $\psi_B = -5\sin\phi\sin\theta$ at $t \approx 36$. The patch centroid was initially at $\phi = \pi/5, \theta = \pi/2$. The barrier has $\theta_0 = 7\pi/20$. The solid line represents the exact point vortex trajectory and the + marks the $(\phi, \theta) = (0, 0)$ location.

irrotational velocity field with streamfunction ψ such that $d(\psi + \tilde{\psi}) = 0$ on $\partial\mathcal{D}$. This harmonic problem can be sterographically projected on to the z-plane such that $\partial\mathcal{D}$ is mapped to $\partial\mathcal{D}_z$ where $z = re^{i\alpha}$ with $\alpha = \phi$ and $r = \cot(\theta/2)$. The boundary $\partial\mathcal{D}_z$ can then be conformally mapped to the domain exterior to the unit circle, here denoted the ζ-plane, such that $\partial\mathcal{D}_z$ is mapped to $\partial\mathcal{D}_\zeta$ where $\zeta = e^{i\hat{\phi}}$ $(0 < \hat{\phi} < 2\pi)$. The velocities owing to $\tilde{\psi}$ upon $\partial\mathcal{D}$ are also appropriately mapped to $\partial\mathcal{D}_\zeta$. Laplace's equation, $\nabla^2_\zeta\psi = 0$, is then solved in the ζ-plane such that the no-normal-flow boundary condition is satisfied. ψ is then conformally mapped back to the unit sphere and the velocity field owing to ψ is added to that due to $\tilde{\psi}$. The vortex patch is then advected and contour nodes redistributed on the surface of the sphere (see Dritschel [2] for details).

3 Examples

Solid boundaries including an ellipse with $b = a$ (a spherical cap) and $b = 0$ (a thin barrier), a longitudinal wedge and half-longitudinal wedge are considered. For all boundaries considered, centroid trajectories of small patches which remain close to circular are in very good agreement with the exact point vortex trajectories. Deviation from these trajectories is seen in cases where the patch is in close proximity to the boundary and is "pinched" against it. In some cases this can then lead to deformation of the patch and filamentation. For the case of the ellipse with $b = 0$ the barrier corresponds to a solid wall around the great circle corresponding to $\phi = 0, \pi$ except for a gap about the north pole corresponding to $0 < \theta < \theta_0$. Compatible background flows prescribed by ψ_B are then superimposed on the velocity field. Figure 1 demonstrates a flow prescribed by $\psi_B = C\sin\phi\sin\theta$ (a flow parallel to the thin barrier), where C is a constant determining the speed of the flow. In the system considered in figure 1 the flow initially opposes the motion of the vortex patch. It is seen that the leading edge of the patch then slowly makes its way around the tip of the barrier. The flow is then assisting the motion of the leading edge of the patch but opposing its rear. The patch is thus split in two and the rear is trapped by the edge of the plate in a quasi-stable state.

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