# ON THE STATICS AND DYNAMICS OF POINT VORTICES IN PERIODIC DOMAINS

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*Summary* The motion of three interacting point vortices with zero net circulation defines an integrable Hamiltonian dynamical system in either singly- or doubly-periodic domains. The singly-periodic case is of interest in the study of 'P+S' wakes behind an oscillating cylinder, and the doubly-periodic case in the study of two-dimensional turbulence. The three-vortex motion in both cases can be reduced to advection of a passive particle by fixed vortices in an equivalent Hamiltonian system. A survey of the solutions for vortex motion in these systems will be discussed. Some initial conditions lead to *relative equilibria*, or vortex configurations that move without change of shape or size. These configurations can be determined as stagnation points in the reduced problem or through explicit solution of the governing equations. A study of relative equilibria of three vortices in doubly-periodic domains will be presented for the first time.

## BACKGROUND

Ever since Helmholtz [4] introduced the concept of what we now call a point vortex, it has been known that the dynamics of two such vortices in the unbounded plane produces only relative equilibria, with the separation between the vortices remaining constant for all time. If the net strength of the vortices is zero, they translate uniformly in a direction perpendicular to their separation; if instead the net strength is non-zero, they rotate about the center of vorticity, which lies on the line passing through both vortices. When the system consists of *three* vortices in the plane, the dynamics remains integrable but the vortex trajectories are more complex, with most initial conditions leading to time dependent vortex separations. The general three-vortex system in the plane has been investigated quite thoroughly (see, e.g., [1]).

The investigation of point vortices in singly-periodic domains began with von Kármán's [5] model of the laminar wake behind a bluff body, although technically he considered an infinite number of vortices in the unbounded plane. This flow can be modeled as two oppositely-signed point vortices in a periodic strip, i.e. in a singly-periodic domain, in which case every choice of vortex separation leads to a relative equilibrium. If instead the net circulation is non-zero the behavior is still integrable and relatively straightforward [7], but most initial conditions do not lead to relative equilibria. Some motivation for considering systems with three or more vortices per period comes from experimental observations of the 'P+S' mode in the wake behind an oscillating cylinder [9].

Doubly-periodic vortex configurations were first considered by Tkachenko [8], who examined the simple lattice consisting of only one vortex and its periodic images. Until recently, investigations of point vortices in periodic domains have focused primarily on the simple lattice. Some motivation for considering systems with three or more vortices per period comes from an interest in reduced-order modeling of vortices in two-dimensional turbulence.

## VORTEX DYNAMICS IN PERIODIC DOMAINS

The equations of motion for $N$ interacting point vortices in a periodic domain are, in complex variable form,

$$\frac{\mathrm{d}\overline{z}_\alpha}{\mathrm{d}t} = \frac{1}{2\pi\mathrm{i}} \sum_{\beta=1}^{N}{}' \Gamma_\beta \, \Phi(z_\alpha - z_\beta), \tag{1a}$$

where $z_\alpha = x_\alpha + \mathrm{i}\, y_\alpha$ is the position of a vortex and $\Gamma_\alpha$ its strength, the overbar denotes complex conjugation, the prime on the sum indicates omission of the singular term $\alpha = \beta$, and the functions for a singly-periodic domain ($\Phi_{\text{cyl}}$) and a doubly-periodic domain ($\Phi_{\text{tor}}$) are given by [2, 6, 3]

$$\Phi_{\text{cyl}}(z) = (\pi/L)\cot[\pi z/L], \tag{1b}$$

and

$$\Phi_{\text{tor}}(z) = \zeta(z;\omega_1,\omega_2) + [(\pi\overline{\omega}_1)/(\Delta\omega_1) - \eta_1/\omega_1]\, z - \pi\overline{z}/\Delta, \tag{1c}$$

respectively. In (1) $L$ is the period of the cylinder, $\zeta(z;\omega_1,\omega_2)$ is the Weierstrass zeta function defined on a torus with half-periods $\omega_1$ and $\omega_2$, and $\Delta$ is the area spanned by the periods $(2\omega_1, 2\omega_2)$. The focus of this discussion will be on systems governed by (1) with $N = 3$ and net zero circulation, in which case the dynamics can be represented by advection of a passive particle at $Z = z_1 - z_2$ by a certain set of fixed vortices, as shown for example in Fig. 1. Solutions for the dynamics with $N = 3$ and $S = 0$ have been presented in the literature for both singly-periodic [2] and doubly-periodic [6] domains, and these results will be surveyed.

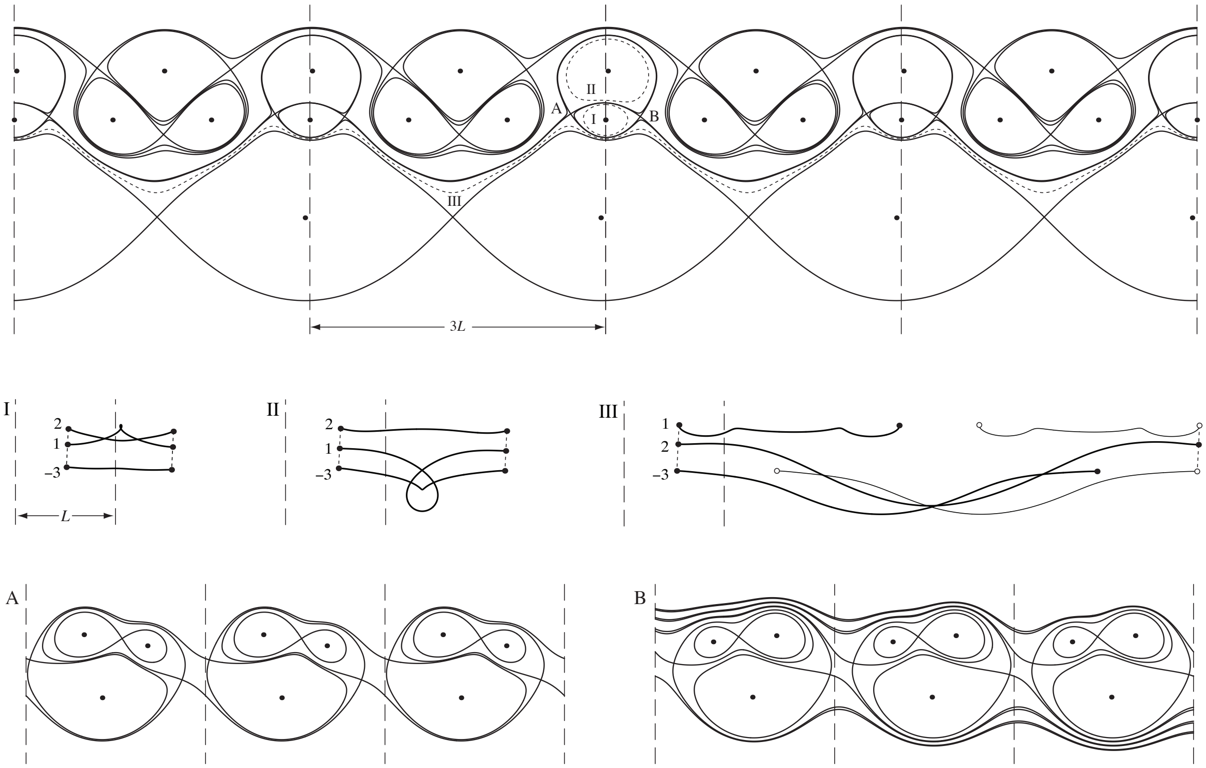


**Figure 1.** *(top)* Separating streamlines in the related advection problem for three vortices with strengths $(2, 1, -3)$ and a specified linear impulse in a periodic strip of width $L$. *(center)* Representative trajectories of the three vortices, labeled according to the corresponding regime in the advection problem. *(bottom)* Co-moving streamlines in two of the relative equilibrium configurations.

## VORTEX STATICS IN PERIODIC DOMAINS

There are a number of vortex configurations in periodic domains that form relative equilibria. In the case $N = 3$ and $S = 0$, these equilibria correspond to stagnation points in the equivalent advection problem discussed above, as illustrated in Fig. 1. Equilibria can also be found by looking explicitly for configurations in which each vortex moves with a constant velocity $V$. Using the notation

$$\Phi_1 = \Phi(z_2 - z_3), \quad \Phi_2 = \Phi(z_3 - z_1), \quad \text{and} \quad \Phi_3 = \Phi(z_1 - z_2), \tag{2a}$$

the equations governing the relative equilibria become

$$\Phi_1 + \Phi_2 + \Phi_3 = 0. \tag{2b}$$

Given one of the vortex separations, combining equation (2) with the appropriate addition formula for either the cotangent or the Weierstrass zeta function gives the other two vortex separations in an equilibrium configuration. The discussion will cover the analysis of relative equilibria in a singly periodic domain [7] and the first presentation of three-vortex relative equilibria in a doubly periodic domain.

### References


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