On the motion of thin vortex tubes

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We consider the motion of a tube of vorticity with circulation Γ and a cross sectional radius σ that is everywhere small compared to local radius of curvature of the tube. In particular, we determine the motion of the space curve C that traces the centerline of the tube for an arbitrary distribution of axial vorticity within the core.

Consideration of special case of a thin vortex ring of radius R begins with Kelvin's (1867) result for the speed of a ring U_R with a uniform distribution of vorticity within the core (see Lamb 1932),

$$U_R = \frac{\Gamma}{4\pi R}\left[\log\frac{8R}{\sigma} - \frac{1}{4} + o\left(\frac{\sigma}{R}\right)\right]. \tag{1}$$

Later Hicks (1884) analyzed the case of a hollow vortex ring with the result that the factor 1/4 in the above formula is replaced by 1/2. In 1970 Fraenkel generalized these results to inviscid rings with an arbitrary distribution of vorticity within the core. In that same year Saffman showed that Fraenkel's analysis could be simplified considerably by transforming a certain integrand in the expression for the energy into an alternate form. This transformation is derived from the Euler equations and was originally attributed to Lamb (1932). However, Shariff & Leonard (1992) noted that the same transformation appears in Helmholtz (1858) with correction.

As in the case of a vortex ring, the straightforward application of the Biot-Savart law to the centerline of a three-dimensional vortex tube leads to a logarithmic divergence for the velocity as $\sigma \to 0$. Over the years various ad hoc procedures have been proposed to account for finite-core effects, typically employing a smoothing procedure with a free parameter that is adjusted to produce the correct result for the speed of a vortex ring. For a discussion see Leonard (1985).

In this paper we extend Saffman's (1970) analysis to determine the equation of motion of the space curve of a thin three-dimensional vortex tube. Briefly, we take the vorticity field to have the representation

$$\boldsymbol{\omega}(\boldsymbol{x}) = \Gamma\int_C \frac{p(|\boldsymbol{x}-\boldsymbol{r}(\xi)|/\sigma)}{\sigma^3}\frac{\partial\boldsymbol{r}}{\partial\xi}d\xi, \tag{2}$$

where $\boldsymbol{r}(\xi)$ defines the space curve with parameter ξ and p, properly normalized, determines the vorticity distribution over the core of the tube. The fluid energy is then given by

$$E = \frac{\Gamma^2}{8\pi}\int_C\int_C \frac{f(|\boldsymbol{r}(\xi)-\boldsymbol{r}(\xi')|/\sigma)}{|\boldsymbol{r}(\xi)-\boldsymbol{r}(\xi')|}\frac{\partial\boldsymbol{r}}{\partial\xi}\cdot\frac{\partial\boldsymbol{r}}{\partial\xi'}d\xi d\xi', \tag{3}$$

where the function f is related to p. Alternatively, the energy may also be written as

$$E = \int \boldsymbol{u} \cdot (\boldsymbol{x} \times \boldsymbol{\omega}) d\boldsymbol{x} \tag{4}$$

or using (2)

$$E = \Gamma \int_C \int \boldsymbol{u}(\boldsymbol{x}) \cdot \left(\boldsymbol{x} \times \frac{\partial \boldsymbol{r}}{\partial \xi} \right) \frac{p(|\boldsymbol{x} - \boldsymbol{r}(\xi)|/\sigma)}{\sigma^3} d\boldsymbol{x} d\xi. \tag{5}$$

Now we decompose the velocity field in the vicinity of the tube as follows,

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{U}(\xi) + \boldsymbol{u}'(\boldsymbol{x}, \xi), \tag{6}$$

where $\boldsymbol{U}(\xi)$ is the velocity of the space curve C and $\boldsymbol{u}'$ is the velocity field relative to $\boldsymbol{U}$. The analysis proceeds by using Helmholtz' transformation on the integrand involving $\boldsymbol{u}'$ and, ultimately, determining $\boldsymbol{U}(\xi)$ such that (3) and (5) produce the same result. Interestingly, the contribution of $\boldsymbol{u}'$ to E in (5) leads to local induction contribution to $\boldsymbol{U}$ while the remaining contribution is given in terms of a Biot-Savart integral over the curve C with a mollifying function depending on the function p of (1).

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