Determining the stability of steady inviscid flows through preferred bifurcation diagrams

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More than a century ago, Lord Kelvin (1875) proposed an energy-based argument for determining the stability of steady inviscid flows. While the key underpinnings of the method are well established, its practical use has been the subject of extensive debate. In this work, we draw on ideas from dynamical systems and imperfection theory to construct a methodology that represents a rigorous implementation of Kelvin's argument. Besides yielding stability properties, which are found to be in precise agreement with the results of linear analysis, our approach also implicitly yields new bifurcated solutions branches, as we shall describe below.

Kelvin's original argument (Kelvin, 1875) states that steady inviscid flows are associated with stationary points of the kinetic energy, for a given linear or angular impulse. For example, for a two-dimensional vortical flow with excess kinetic energy E and angular impulse J, one can construct a functional H:

$$H = E - \Omega J, \tag{1}$$

such that the first variation δH with respect to vorticity-preserving perturbations vanishes if, and only if, the flow is in equilibrium if observed in a frame rotating with angular velocity Ω (see Saffman, 1992). One may then proceed to establish stability of the solution as follows. Since E and J are both conserved in an inviscid flow, H is also a conserved quantity. If the stationary point is a maximum or a minimum in the solution space (implying that the second variation $\delta^2 H$ is positive or negative definite), then a displacement away from the solution would lead to a change in H, which is impossible; hence the solution must be stable to isovortical perturbations, thus yielding a sufficient condition for stability. Similarly, a necessary condition of instability is that the stationary point is a saddle (Kelvin, 1875).

Stability can therefore be established by directly evaluating the second variation of H; this approach carries over to all conservative systems for which a similar functional can be defined (e.g. Holm *et al.*, 1985; Vladimirov *et al.*, 1999). However, implementing this methodology is not always feasible, especially since several steady solutions of practical interest are known only numerically. It is therefore common to resort to computing eigenvalues through a linear stability analysis, which is usually a process far more laborious than computing the steady solutions.

Saffman & Szeto (1980), having numerically found steady solutions for two co-rotating vortices, circumvented this difficulty as follows. Equation (1) can be interpreted as establishing extrema of E under the constraint that $J =$ const., Ω taking the role of a Lagrange multiplier. A plot of E versus J then shows that, for a given J, there exist two E branches, joined at a fold point. The top branch was interpreted as a maximum (and hence stable), while the lower branch as a saddle (possibly unstable). This prediction was subsequently found to match the results of the linear stability analyses of Kamm (1987) and Dritschel (1995). The same approach was later used for several other flows (Saffman & Szeto, 1981; Saffman & Schatzman, 1982).

A fundamental objection to such an implementation of Kelvin's argument was however posed by Dritschel (1985), who pointed out that there seems to be no necessary link between the shape of a plot of E versus J and the curvature of the H surface. That is, one cannot determine whether the highest solution branch in a two-dimensional plot does actually correspond to a maximum in the solution space. Furthermore, Dritschel (1985) pointed out that even if such correspondence could be established, additional changes of stability could also occur away from extrema in E and J by means of bifurcations to new families of solutions. This would render the method unreliable. In a later work, Dritschel (1995) provided an example for which the method fails to correctly predict the onset of instability. He considered the family consisting of two uniform vortices with equal vorticity magnitude, opposite sign, and unequal area, and inspected different area ratios A_1/A_2, for which he computed both equilibrium states and linear stability properties. For families with a A_1/A_2 fixed between 1 and 0.9, the extremum in E gave an accurate prediction for the onset of instability. However, below $A_1/A_2 \simeq 0.9$ the change of stability occurred before the location predicted by a plot of E versus J. This provided definitive proof that such an implementation of Kelvin's argument does not always work.

In the light of such a rich history of developments, we have been stimulated to employ tools from dynamical systems theory to devise a new approach, which provides a rigorous link between extrema in a particular bifurcation diagram and changes in the second variation of a functional such as H. The essential step in the application of the theory involves the identification of two quantities that can be used to construct what we define as a "preferred" bifurcation diagram. For the case in question it emerges that the relevant plot is not one of E versus J, but of J versus Ω. Extrema in J are therefore linked to stability properties.

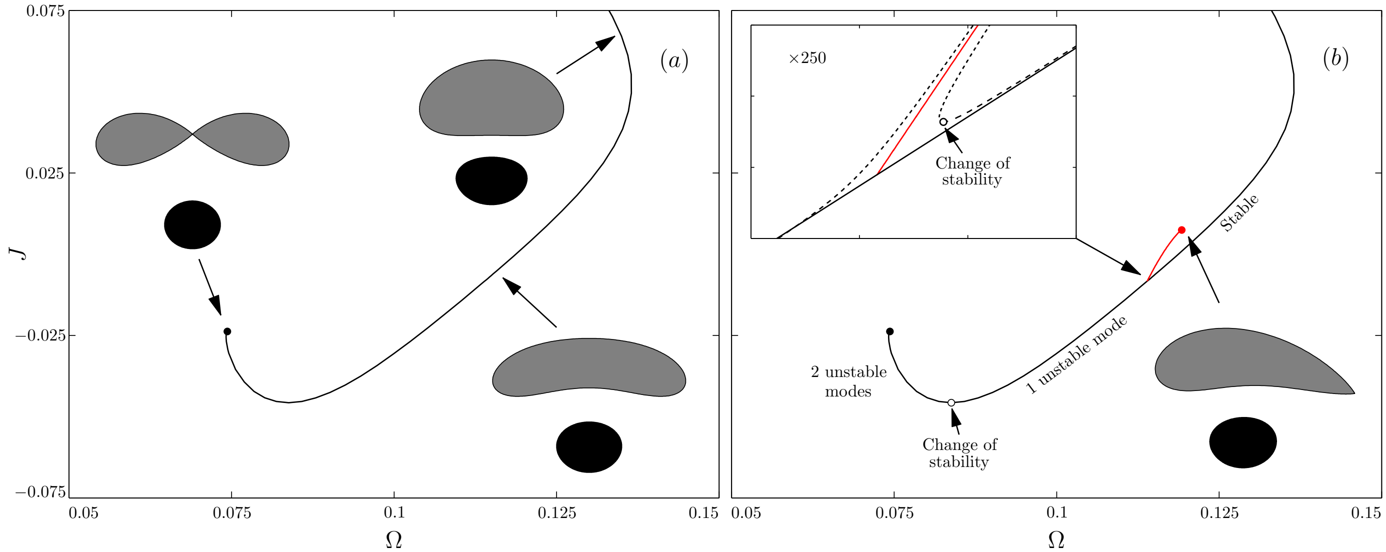

Figure 1: "Preferred bifurcation diagram" for the opposite-signed pair with $A_1/A_2 = 0.3$. The basic family is shown in (*a*). Re-computing the solutions with a small perturbation (dashed lines in (*b*)) yields the stability properties for the family, while revealing a new bifurcated branch (red line). Filled circles denote limiting shapes; empty circles represent changes of stability.

Nevertheless, the second objection posed by Dritschel (1985) would still stand, since, while an extremum in J is always associated with a change in the curvature of the H surface, changes in stability that occur through bifurcations would be undetected (without performing a linear stability analysis). We resolve this issue by employing the fact that joins in solution branches are not structurally stable (e.g. Poston & Stewart, 1978). Hence by slightly perturbing the original system we obtain a new system for which the solution branches will be distinct; thus any bifurcations are uncovered, and all changes of stability are apparent with our methodology.

Eager to verify whether this novel approach would work, we re-examined the opposite-signed family studied by Dritschel (1995). Fig. 1 shows the "preferred" bifurcation diagram J vs Ω for $A_1/A_2 = 0.3$. Introducing a small perturbation and re-computing the equilibria breaks the family of solutions into distinct branches (dashed lines in the inset of fig. 1*b*), hinting at the presence of a bifurcation. Our method then reveals another extremum in J, which is associated with an additional change of stability. It turns out that the location of this change of stability, found using our approach based on preferred bifurcation diagrams, agrees precisely with previous results from linear analysis. Finally, by bringing the perturbation to zero, our approach enables us to determine a new bifurcated branch (red line in fig. 1*b*), revealing a new family of solutions of lesser symmetry, whose vortex shapes are exhibited in fig. 1*b*.

In further work, we have applied the same approach to a wide range of classical solutions of the Euler equations, including for example the Kirchhoff elliptical vortices, the co-rotating vortex pair and its continuation into a singly connected shape (Cerretelli & Williamson, 2003), the finite-area Kármán street (Saffman & Schatzman, 1982), Stuart vortices (Pierrehumbert & Widnall, 1982), and other flows. For all cases considered, we find precise correspondence with classical results from linear analysis, while additionally discovering new families of steady solutions.

We would like to acknowledge several useful discussions with Prof. S. Leibovich on the subject.

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