Vortices on closed surfaces

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One hundred and fifty years after Helmholtz' "Wirbel" paper [7], the study of vortices on surfaces is still in its infancy, confined basically to the sphere or surfaces of revolution[3]. As far as we know an intrinsic Hamiltonian formulation for the motion of N point vortices s_j of strengths κ_j on a closed (compact, boundaryless, orientable) surface S with riemannian metric g is in order. We hope to fill this gap, providing also some clarifications to [6]. For the full version of this note, see arXiv:0802.4313v1 [math.SG].

1. *Desingularizing the stream function is mainstream mathematics (no pun intended).* The stream function produced by a unit point vortex at s_o on a background uniform counter vorticity is given by Green's function $G_g(s, s_o)$ of the Laplace-Beltrami operator $\Delta_g = \text{div}g \circ \text{grad}_g$[4]. For any surface, G_g behaves near s_o as $\log d(s, s_o)/2\pi$ [12]. An energy core argument [5] implies that a single vortex s_o drifts on S according to the Hamiltonian system (Ω_g, R_g), where Ω_g is the area form of g and R_g is Robins's function $R_g(s_o) = \lim_{s \to s_o} G_g(s, s_o) - \log d(s, s_o)/2\pi$, an important object form geometric function theory. Moreover, if S has genus zero, it it known that $R_g = \Delta_g^{-1} K + \text{trace}\Delta_g^{-1}/A(S)$.

2. *The collective Hamiltonian keeps the same form of C.C.Lin seminal paper* [10],

$$\Omega_{\text{collective}} = \sum_{j=1}^{N} \kappa_J \Omega(s_j) \ \ , \ \ H = \sum_{1 \le i < j \le N} \kappa_i \kappa_j G_g(s_i, s_j) + \sum_{\ell=1}^{N} \frac{1}{2} \kappa_\ell^2 R_g(s_\ell) \ .$$

[For *Jordan domains* $D \subset S$ the structure is the same, using the appropriate hydrodynamical Green function. An extension for *vortices with mass* is also immediate.]

3. *Under conformal changes of metrics* $\tilde{g} = h^2 g$ the symplectic form changes accordingly to $\Omega_{\tilde{g}} = h^2 \Omega_g$ while the new Hamiltonian is given by

$$\tilde{H} = H - \frac{1}{4\pi} \sum_{\ell=1}^{N} \kappa_\ell^2 \, \log(h(s_\ell)) - \frac{\kappa}{\tilde{A}(S)} \sum_{\ell=1}^{N} \kappa_\ell \Delta^{-1} h^2(s_\ell) \ \ , \ \ \kappa = \sum_{\ell=1}^{N} \kappa_\ell \ .$$

[The presence of the total vorticity reflects the fact that when it vanishes, the collective vortex stream function $\psi(s; s_1, ..., s_N)$ is independent of the conformal metric $\tilde{g} = \exp(2\phi)g$. In this case the $\tilde{g}$-regularization of ψ at any of the vortices simplifies: one just subtracts off $\frac{1}{2\pi}\phi(s_j)$ from the g-regularized stream function at s_j. In particular, when S is conformal to the standard sphere, making an artificial puncture at any given $s^* \in S$ allows to easily write vortex motions for any metric on the sphere.]

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[3] See [1], [3], [11] for reviews and a treatise on point vortices

[4] Bogomolov [4] and Kimura/Okamoto [8] already used them for the sphere S^2.

4. *We give a simple proof of Kimura's conjecture: a vortex dipole describes geodesic motion.* Therefore, searching for integrable vortex pair systems on Liouville surfaces [9] is a natural mathematical question.

6. *Ongoing work.* We are presently focusing on the vortex pair problem, an Hamiltonian system on $S \times S$, where the symplectic form is the difference $\Omega = \kappa(\omega(s_1) - \omega(s_2))$, ω being the area form of S and Hamiltonian given by $H = \kappa^2(-\log d(s_1, s_2)/2\pi + B(s_1, s_2))$, $B(s_1, s_2) = (R(s_1) + R(s_2))/2 - (G(s_1, s_2) - \log d(s_1, s_2)/2\pi)$. We call B Batman's function, seemingly an yet unexplored object. It vanishes on the diagonal and is $O(d(s_1, s_2)^2)$. A more detailed proof of Kimura's conjecture, aiming at perturbation studies, requires its expansion and a blow-up at the diagonal of $S \times S$, transforming the problem to a neighborhood of the zero section of T^*S. We show how the geodesic dynamics in T^*S is perturbed when the two opposite vortices s_1, s_2 are at a small nonzero distance $d(s_1, s_2) = O(\epsilon)$. If the vortex system is integrable, so is the limiting geodesic motion. Integrable vortex pair systems must belong to Liouville surfaces. For which of them does the converse hold? The converse is clearly true for surfaces of revolution, although we verify that the second integrals are unrelated. The vortex system near the diagonal can be represented as a Hamiltonian system (Ω, H) on T^*S with $\Omega = \Omega_{\text{can}} + \epsilon^2 \Omega_1 + ...$ and $H = H_o + \epsilon^2 H_1 + ...$, where (Ω_o, H_o) is the geodesic dynamics. Our current work involves finding the explicit formulas for the deformation term Ω_1 and perturbation H_1 that can be interpreted in terms of geometric quantities. If S is a Liouville surface possessing a homoclinic lagrangian submanifold, one can apply a Melnikov test for non-integrability. The triaxial ellipsoid $E : \sum_{j=1}^3 x_j^2/a_j^2 = 1$ is a perfect arena for experimentation. Jacobi showed in 1838 that the geodesic problem is integrable. To investigate the vortex motion, we can represent the ellipsoid over the sphere $S^2 : \sum_{j=1}^3 \gamma_j^2 = 1$ via the dilations $\gamma_j = x_j/a_j$, and use sphero-conical coordinates in S^2 (rather than confocal quadric coordinates on E). The conformal factor to the sphere metric can be given in terms of elliptic functions. We plan to report on numerical investigations on thems on a subsequent work. Not in our wildest dreams (but why not?) is the possibility that quantizing a vortex system on a surface could relate with a million dollars question [2].

References

[1] Aref, H., Point vortex dynamics: A classical mathematics playground, J. Math. Phys. **48**:6, 065401 (2007)

[2] Berry, M. V., Keating, J. P., The Riemann zeros and eigenvalue asymptotics, SIAM Review **41**, 236–266 (1999)

[3] Boatto, S., Crowdy, D., Point-vortex dynamics, in *Encyclopedia of Mathematical Physics*, ed. by Irina Arefeva, Daniel Sternheimer, Springer-Verlag, to appear.

[4] Bogomolov, V.A., Dynamics of vorticity at a sphere, Fluid Dyn. **6**, 863 – 870 (1977)

[5] Flucher, M., Gustafsson, B., Vortex motion in 2-dimensional hydrodynamics, energy renormalization and stability of vortex pair, TRITA preprint series (1997)

[6] Hally, D., Stability of streets of vortices on surfaces of revolution with a reflection symmetry, J. Math. Phys. **21**:1, 211–217 (1980)

[7] Helmoltz, H., Über integrale der hydrodynamischen gleichungen welche den Wirbelbewegungen entsprechen, Crelles J. **55**, 25-55 (1858) downloadable from http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?did=D268537

[8] Kimura, Y., Okamoto, H., Vortex motion on a sphere, J. Phys. Soc. Jps. **56**, 4203–4206 (1987)

[9] Kiyohara, K, Compact Liouville surfaces, J. Math. Soc. Japan, **43**:3, 555-591 (1991)

[10] Lin, C.C., On the motion of vortices in two dimensions - I. Existence of the Kirchhoff-Routh function; II - some further investigations on the Kirchhoff-Routh function, Proc. Nat. Ac. Sci. , **27**, 570-577 (1941)

[11] Newton, P., *The N-vortex problem. Analytical technques*, Springer, 2001.

[12] Okikiolu, K., A negative mass theorem for the 2-torus, ArXiv [math-SP]: 07113489v1 (2007)