QUASI-GEOSTROPHIC SINGULAR VORTICES EMBEDDED IN A REGULAR FLOW

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New results in the theory of quasigeostrophic (QG) singular vortices in one- and two-layer fluids are presented. On the *f*-plane, the singular vortex is a conventional point vortex, whereas on the β - plane the intrinsic vorticity of singular vortex contains an exponentially decaying term in addition to delta-function. The theory describes singular vortices interacting with each other and with some regular flow. Equations governing the joint evolution of the vortices and the regular field are set up, and the integrals expressing conservation of enstrophy, energy, momentum, and mass are derived. Using these integrals, the stability of a point-vortex pair on the *f*-plane is analyzed. Such vortex pairs are shown to be nonlinearly stable with respect to any small perturbation provided its regular-flow energy and enstrophy are finite. On the β - plane, a new steady exact solution is presented, a hybrid regular-singular modon. The modon consists of a regular dipole component and an axially symmetric rider which is a superposition of the singular vortex and a regular axially symmetric field. The unsteady drift of an individual singular vortex on the β - plane is considered. Whereas the barotropic and baroclinic modes of the singular vortex are comparable in magnitudes, the baroclinic part of β -gyres attenuates with time, making the vortex trajectory close to that of a barotropic monopole on the β -plane.

Outline of work

The model is based on the well-known equations of conservation of QG potential vorticity:

$$\partial q_i/\partial t + J(\psi_i, q_i) + \beta\partial\psi_i/\partial x = 0, \qquad q_i = \nabla^2\psi_i + F_i(a_i\psi_2 - b_i\psi_1), \quad i = 1,2, \qquad \text{(1a,b)}$$

where subscripts "1", "2" correspond to the upper and lower layers, respectively, ψ_i and q_i are the streamfunction and the intrinsic vorticity (IV) in *i*-th layer, respectively, and the constant coefficients F_i, a_i, b_i are determined by the model parameters. Each streamfunction ψ_i is a sum of a regular component $\psi_{i,r}$ and a singular component $\psi_{i,s}$ which represents a superposition of M_i singular vortices having the constant amplitudes A_{i,m_i} and moving along the trajectories $\mathbf{r} = (x_{m_i}(t), y_{m_i}(t))$. The functions $\psi_{i,s}$ are determined by the equations:

$$q_{i,s} - p^2\psi_{i,s} = \sum_{m_i}^{M_i} A_{i,m_i}\delta(x - x_{m_i})\delta(y - y_{m_i}), \quad i = 1,2, \qquad (2)$$

where *p* is a positive constant and q_{is} are IVs (1b) with ψ_{is} instead of ψ_i. If $p = 0$ then the singular vortices are the point ones; for $p \neq 0$ each the vortex contains an additional logarithmic singularity. Joint evolution of the singular vortices and the regular field is governed by the following equations:

$$\partial(q_{i,r} + p^2\psi_{i,s})\big/\partial t + \beta\,\partial(\psi_{i,r} + \psi_{i,s})\big/\partial x + J(\psi_{i,r} + \psi_{i,s}, q_{i,r} + p^2\psi_{i,r}) = 0, \qquad \text{(3a)}$$

$$\dot{x}_{m_i} = -\partial(\psi_{i,r} + \psi_{i,s}^{m_i})\big/\partial y\Big|_{\mathbf{r}=\mathbf{r}_{m_i}}, \quad \dot{y}_{m_i} = \partial(\psi_{i,r} + \psi_{i,s}^{m_i})\big/\partial x\Big|_{\mathbf{r}=\mathbf{r}_{m_i}}, \quad i = 1,2. \qquad \text{(3b,c)}$$

Here $\psi_{i,s}^{m_i}$ is the streamfunction $\psi_{i,s}$ without the m_i -th vortex; equation (3a) describes the regular component and equations (3b,c) - motion of the singular vortices.

System (3) conserves mass, momentum, energy, and enstrophy. The enstrophy integral Ω and the energy integral *E* are of special interest:

$$\Omega = S_r + p^2K_S - \beta\Big(\sum_{m_1} A_{1,m_1}y_{m_1} + \alpha\sum_{m_2} A_{2,m_2}y_{m_2}\Big) = const, \qquad (4)$$

$$E = K_S + E_r - p^2E_{s,r} - \sum_{m_1} A_{1,m_1}\psi_{1r}\Big|_{\mathbf{r}=\mathbf{r}_{m_1}} - \alpha\sum_{m_2} A_{2,m_2}\psi_{2,r}\Big|_{\mathbf{r}=\mathbf{r}_{m_2}} = const. \qquad (5)$$

Here $\alpha = -F_1 a_1 / F_2 b_2$, and E_r and S_r are the energy and enstrophy of the regular component, respectively,

$$E_r = -(1/2)\int(\psi_{1,r} q_{1,r} + \alpha\psi_{2,r} q_{2r})d\mathbf{r}\,, \quad S_r = (1/2)\int(q_{1,r}^2 + \alpha\, q_{2,r}^2)d\mathbf{r} + p^2 E_r\,. \qquad \text{(6a,b)}$$

The sum of the last three terms in (5) is equal to the energy of interaction between the singular and regular components, and

$$E_{sr} = \int(\psi_{1,s}\psi_{1,r} + \alpha\psi_{2,s}\psi_{2,r})d\mathbf{r} + (1/2)\int(\psi_{1,s}^2 + \alpha\psi_{2,s}^2)d\mathbf{r}\,. \qquad \text{(7)}$$

The positive function K_S is the energy of interaction between singular vortices; in general case K_S is represented by a rather cumbersome formula and is not given here.

The enstrophy and energy integrals (4), (5) were used for the analysis of stability of a point vortex pair on the f–plane, when $p = \beta = 0$. In the absence of regular component such a pair either rotates with uniform angular speed or uniformly translates along some straight line, depending on the vortex signs and magnitudes. This motion is stable in the sense that small changes in the distance between the vortices result only in small changes of the pair velocity. Invariants (4) and (5) enable us to prove that the vortex pair is stable with respect to any localized and sufficiently small perturbation. The key point is that on the f - plane the regular IVs, $q_{i,r}$, are conserved in the fluid elements in accordance with (3a), therefore the regular enstrophy S_r is invariant. The estimates $|\psi_{i,r}| < C_1\sqrt{S_r}$, $|E_r| < C_2 S_r$ can be shown to hold, where C_1, C_2 are some coefficients that depend on the model parameters. Thus, if the enstrophy S_r is small at the initial moment, then the regular streamfunctions $\psi_{i,r}$ and the energy E_r remain small at subsequent times. By virtue of (5), this means that variations of the singular energy K_S are also small. In the case of two singular vortices, we get $K_S = G(r_{1,2})$, where $r_{1,2}$ is the distance between the vortices, and the function $G(z)$ is monotonic. Therefore, the smallness of variations of K_S means the smallness of variations of $r_{1,2}$, i.e. the stability of the vortex pair.

If amplitudes and coordinates of the singular vortices are set in a proper way, then the regular field is zero, and the vortices form a system moving along a latitude circle $y = const$ at a constant speed lying outside the range of the phase speeds of linear Rossby waves (for $\beta \neq 0$). Any system of such a kind is a discrete two-dimensional Rossby modon and, vice versa, any distributed Rossby modon is a superposition of the singular vortices concentrated in the interior region of the modon. Here we present a new two-layer modon solution consisting of a singular rider of an arbitrary amplitude driven by a regular dipole component. An important feature of this solution is that both the rider and the regular component are continuous up to the second derivatives (of course, excluding the point of the vortex singularity). Numerical experiments show that smooth modons can be stable or, at least, long-lived.

Dynamics of an individual intense localized vortex on the β-plane were widely discussed in the literature, and by now the evolutions of purely barotropic and purely baroclinic vortices are studied rather well. These two cases are very different. A purely barotropic vortex moves north- or southwestward depending on the vortex sign, and a purely baroclinic vortex tends to become a modon propagating eastward. Here we examine the evolution of an individual singular vortex confined to one of the layers. The barotropic and baroclinic components of such a vortex are comparable in magnitudes. Like the purely barotropic and baroclinic cases, the vortex produces a secondary dipole circulation (the so-called beta-gyres) owing to the beta-effect and non-linearity. The barotropic part of the beta-gyres forces the vortex to move westward along some curved trajectory, and the baroclinic part tends to incline the vortex's axis forcing the vortex to move eastward. The development of the beta-gyres is analyzed and the singular vortex drift velocity is calculated. The main result is that in the presence of a sufficiently strong barotropic mode in the initial vortex the baroclinic part of beta-gyres tends to zero with increasing time, so that the motion of the "mixed mode" vortex becomes similar to that of a barotropic vortex.

Acknowledgements
The study was supported by the RFBR Grants 05-05-64212 and 08-05-00006.