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Stability of classical flows and new vortical solutions from preferred bifurcation diagrams

Charles H. K. Williamson and Paolo Luzzatto-Fegiz

Sibley School of Mechanical and Aerospace Engineering,
Cornell University

In a related research contribution (which has been submitted for a lecture presentation at this symposium), we present a methodology, based on ideas from dynamical systems and imperfection theory, which provides a rigorous and practical implementation of Kelvin's variational argument to determine the stability of inviscid flows. In this poster submission, we intend to illustrate in further detail the application of our methodology by considering a wide range of classical flows. For all cases studied, the results from our method correspond precisely with predictions from linear analysis, while additionally discovering new steady solutions. In the present abstract, we show one example out of the set of classical flows we shall present at the conference.

Lord Kelvin (1875) proposed that, for isovortical perturbations, steady inviscid flows are stationary points of the kinetic energy E, for a given impulse of the vortex configuration. For example, a vortical structure steadily rotating at a rate Ω and having impulse J represents a stationary point of $H = E - \Omega J$. Since E and J are conserved, Kelvin argued that the second variation $\delta^2 H$ can constrain the evolution of the system and thus can be used to assess stability.

Saffman & Szeto (1980) suggested that explicit computation of $\delta^2 H$ could be avoided by instead using extrema in a diagram of E vs J to identify changes in $\delta^2 H$. However, Dritschel (1985) later pointed out that no firm link had been established between $\delta^2 H$ and a diagram of of E vs J. Furthermore, he stated that even if such connection could be proven, changes of stability could also occur, through bifurcations, at locations away from extrema in E and J. Dritschel therefore conclusively argued that such an implementation of Kelvin's argument would not always work.

We address both of these issues by proposing a new approach to this problem. We introduce a theorem from dynamical systems theory to prove that extrema of J are indeed related to the properties of $\delta^2 H$, and that the relevant plot is one of J vs Ω (instead of E vs J). Dritschel's second objection is resolved by exploiting the fact that joins in solution branches are not structurally stable. By introducing a small perturbation and re-computing the steady states we are therefore able to obtain distinct solution branches, thus uncovering any bifurcations.

The last point is better illustrated by considering, as one of the examples we shall present in our poster, the stability of the family of solutions found by Kirchhoff, namely uniform-vorticity elliptical patches (e.g. Lamb, 1932). Under a suitable nondimensionalization, it can be shown that:

$$J(\Omega) = \frac{2\Omega - 1}{8\pi\Omega},$$

where Ω is related to the axis ratio $\lambda = b/a$ by $\Omega = \lambda(\lambda+1)^{-2}$. Since J is singular as $\lambda \to 0$, and $J = (4\pi)^{-1}$ at $\lambda = 1$, we choose to plot $-(4\pi J)^{-1}$ instead of J (see fig. 1; notice that the value of Ω for any extrema would be unchanged). Since $J(\Omega)$ is monotonic, any changes of stability have to occur through bifurcations. It would therefore seem impossible to determine stability properties from a plot of J vs Ω, in this particular example.

Fig. 2 shows the result of applying our approach involving imperfection theory; the underlying procedure is illustrated in fig. 2*b*, where the first bifurcation that we find (which is associated with a deformation with azimuthal wavenumber $m = 3$) is considered. Introducing a small perturbation and re-computing the equilibria numerically, we

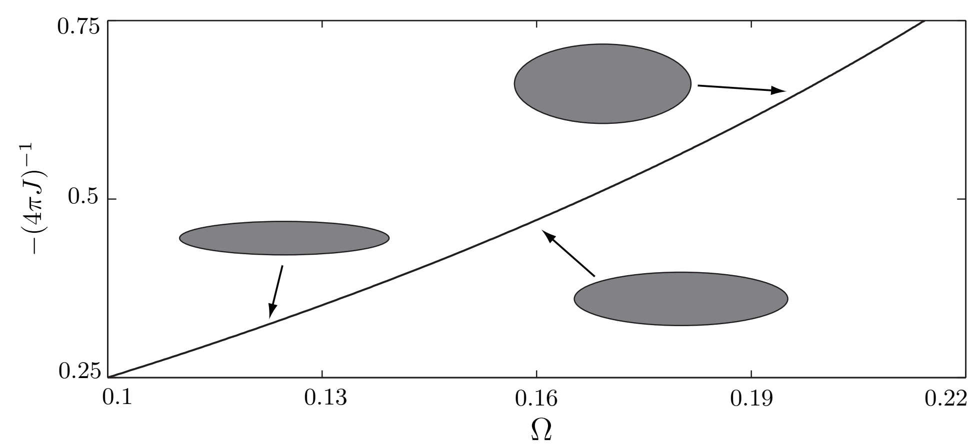

Figure 1: Plot of angular impulse versus angular velocity for the Kirchhoff ellipses.

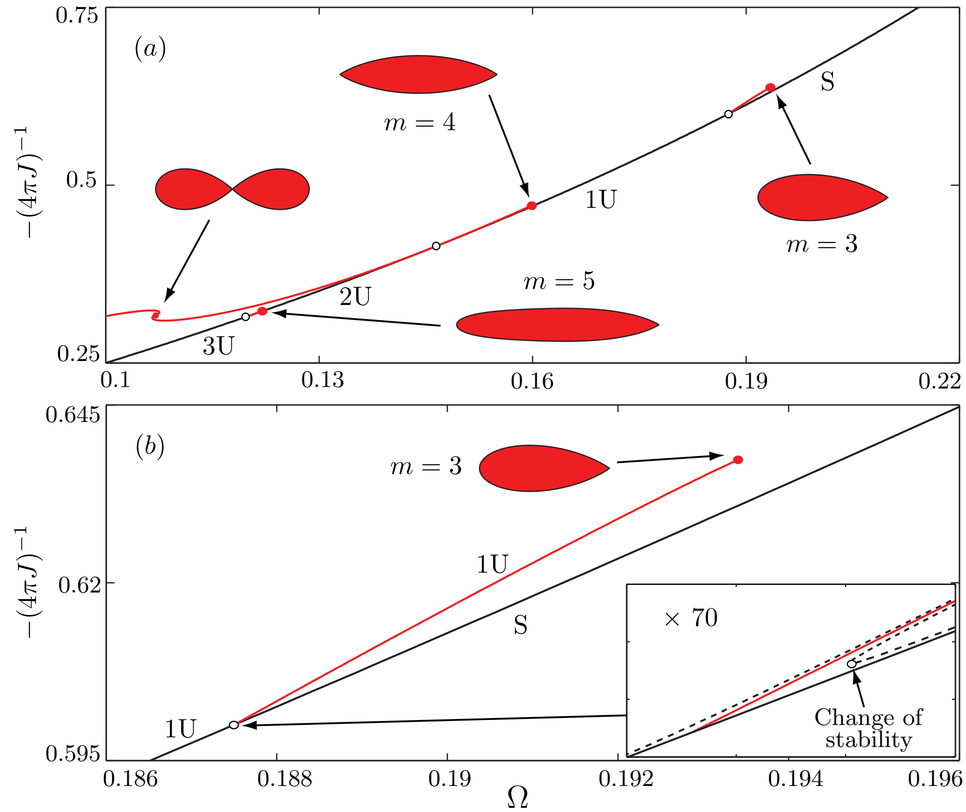

Figure 2: The diagram resulting from the application of our method is shown in (*a*). In (*b*), the first ($m = 3$) branch is shown in detail. Re-computing the solutions with a small perturbation (dashed lines) yields stability properties, while revealing new bifurcated branches (red lines). Open circles denote changes of stability; filled red circles represent limiting shapes. 'S' stands for 'stable'; 'nU' denotes instability to n modes.

break the original curve into distinct branches (dashed lines in the inset), revealing an otherwise undetected change of stability at a minimum in J (illustrated by the inset in fig 2*b*). By bringing the perturbation to zero, we recover a bifurcated branch of lesser symmetry (displayed in red), which we then follow numerically up to a limiting shape (shown in fig. 2*b*). Applying the same approach to the rest of the family (fig. 2*a*), we proceed to identify the next two bifurcations, which are associated with deformations with $m = 4, 5$. While the $m = 3$ and $m = 5$ bifurcations are of subcritical type, $m = 4$ is transcritical. All of these bifurcated branches are found to terminate with limiting shapes (shown in fig. 2*a*), which upon close inspection are seen to exhibit one or more 90° corners.

Finally, it is reassuring to observe that the locations of the changes of stability found through our methodology precisely match the prediction of the linear analysis of Moore & Saffman (1971), which formed the basis of the work of Kamm (1987), who computed the beginning of these bifurcated branches. While the $m = 4$ branch was later explored in its entirety by Cerretelli & Williamson (2003), it appears that the limiting states for the $m = 3, 5$ cases had not been computed before.

We have applied our procedure to several other steady solutions, including the co-rotating pair and its continuation into a singly connected shape (Cerretelli & Williamson, 2003), the counter-rotating pair (Dritschel, 1995), the finite-area Kármán street (Saffman & Schatzman, 1982), Stuart vortices (Pierrehumbert & Widnall, 1982), and other flows.

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