# Study of Vortex Flows behind the Circular Cylinder

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The investigation of vortex flows behind the cylinder (cylinder wake) is conducted numerically by use of the spectral method. We newly developed the method which can deal with the degree of freedom of $O(10^5)$ owing to the way of searching for the fixed point of the Poincaré return map with the aid of the GMRes($k$) method, and can treat an infinite flow field around the cylinder without using the radiation boundary condition. Therefore a detailed study of cylinder wake is possible up to $Re = O(10^3)$, where Reynolds number $Re$ is defined by the diameter of the cylinder $d$, far field flow velocity $U$ and kinematic viscosity $\nu$ as $Re = dU/\nu$. As a remarkable result, we found plural solutions for two-dimensional (2D) periodic flow (von Kármán vortex street) as well as a steady twin-vortex flow between $177 < Re < 188$ (see Fig. 1). It is known that the critical Reynolds number where 2D flow meets three-dimensional (3D) instability is around $Re = 180$.

For the cylinder wake, a lot of studies have been conducted for a long time. Among them, Zebib's pioneering work[1)] should be mentioned at first. He addressed the stability of steady solutions by use of the spectral method and found that the twin-vortex steady solution becomes unstable at $Re_c \approx 40$. This work is succeeded by Noack and Eckelmann [2, 3)] in 1990s. They investigated the stability of steady and periodic solutions by use of the low-dimensional Galerkin method and found that the critical Reynolds number of the steady solution $Re_{crit1} = 54$ and that of the 2D periodic one $Re_{crit2} = 170$. However, the degree of freedom they used was extremely small. Thus, it is expected to study this problem with large number of freedom.

The basic equation, 2-D vorticity equation

$$\frac{\partial}{\partial t}\left(\nabla^2\tilde{\psi}\right) = \frac{1}{r}\frac{\partial(\tilde{\psi}, \nabla^2\tilde{\psi})}{\partial(r,\theta)} + \frac{2}{Re}\nabla^2\nabla^2\tilde{\psi} \tag{1}$$

is solved under the boundary condition

$$\tilde{\psi} = \psi_c(t)\ (r=1), \quad \tilde{\psi} = r\sin\theta\ (r\to\infty) \tag{2}$$

where $\tilde{\psi}(r,\theta,t)$ is the stream function, $r$ and $\theta$ are variables in the radial- and azimuthal-direction and $\psi_c(t)$ denotes an indefinite constant which will be determined by the condition that pressure be single-valued on the cylinder surface. $\tilde{\psi}$ is expressed by the sum of two components as $\tilde{\psi}(r,\theta,t) = \bar{\psi}(r,\theta) + \psi(r,\theta,t)$.

To solve $\psi$ numerically, $\psi$ is expanded in the Fourier series as

$$\psi(r,\theta,t) = \sum_{n=-\left[\frac{N}{2}\right]}^{\left[\frac{N-1}{2}\right]} \mathrm{e}^{in\theta}\psi_n(r(x),t), \tag{3}$$

where $N$ is the truncation number. Further, $r$ is mapped to the finite region $x$ by

$$x = 1 - 2\mathrm{e}^{c(1-r)}\ \ (-1 \le x \le 1), \tag{4}$$

where $c$ is a scale factor. Then $\psi_n$ is expanded in the $x$-direction and $\psi$ is expressed as

$$\psi(r,\theta,t) = \sum_{n=-\left[\frac{N}{2}\right]}^{\left[\frac{N-1}{2}\right]} \mathrm{e}^{in\theta}\sum_{m=1}^{M}\Psi_m^{(n)}(r(x))\psi_{mn}(t), \tag{5}$$

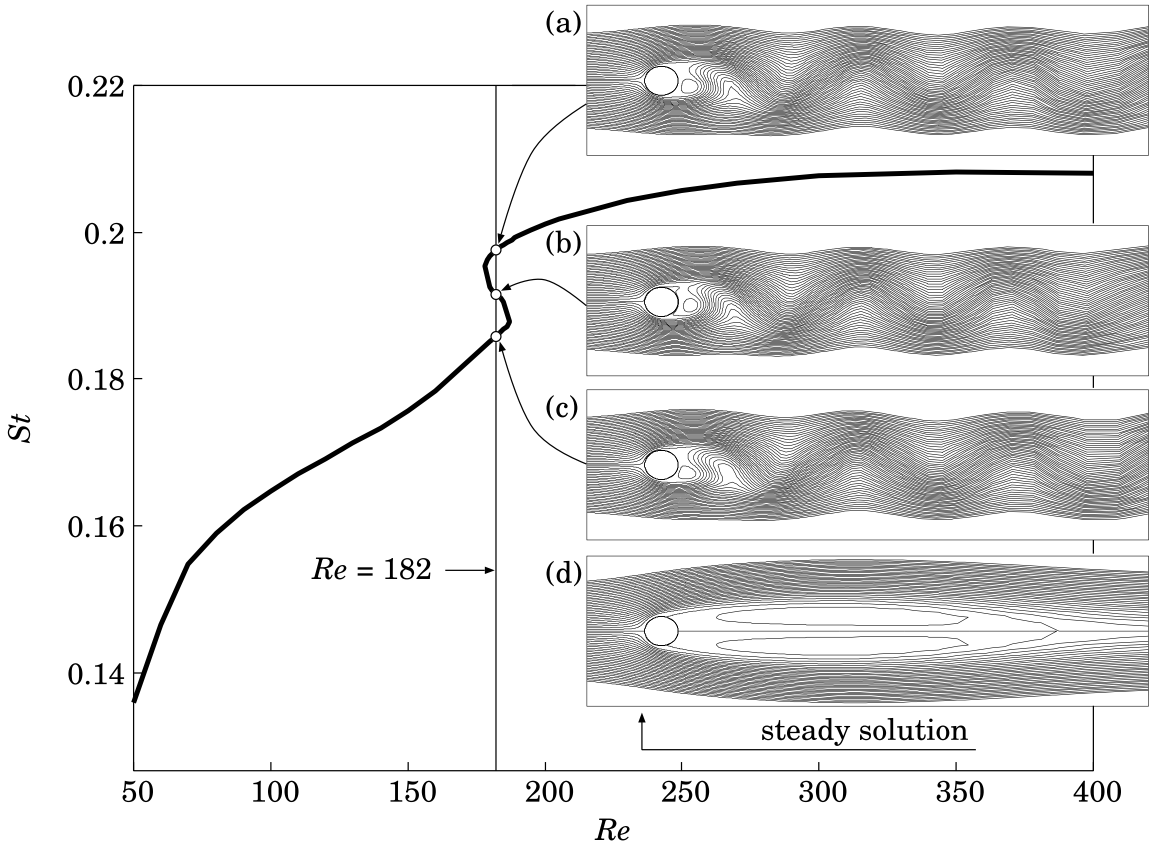


Fig. 1. Bifurcation diagram of periodic solutions and four different solutions at $Re = 182$.

where $M$ is the truncation number and $\Psi_m^{(n)}(r(x))$ is defined by use of the Chebyshef polynomials $T_m(x)$ modified as to satisfy the boundary condition on the cylinder surface. Now expansion coefficient $\psi_{mn}$ is solved numerically by applying of the collocation method.

Utilizing numerical method introduced above, time-evolution calculations are performed by use of the forward time difference scheme together with the Adams-Bashforth and Crank-Nicolson methods. Steady solutions are calculated utilizing the Newton-Raphson iteration method and periodic solutions are obtained searching for the fixed point in the corresponding Poincaré return map combined with the GMRes($k$) method. Linear stability and Floquet analysis are applied to investigate the stability of the steady and periodic solutions, respectively. As the results of calculations, periodic solutions for various $Re$ are obtained. The bifurcation diagram and flow patterns at $Re = 182$ are shown in Fig. 1. The ordinate $St$ is the Strouhal number defined by the period of the periodic solution $T$ as $St = 2/T$. In the figure, (a), (b) and (c) are flow patterns of periodic solutions which have different Strouhal numbers with each other and (d) is that of a steady twin-vortex solution.

A great efficiency is brought about by utilizing the GMRes($k$) method with the calculation searching for the fixed point in a Poincaré section. The analysis of cylinder wakes for higher Reynolds numbers is under investigation. It is expected that chaotic vortex motions can be understood in the light of the structure of coherent vortex motions in unstable periodic solutions.

# References

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