Sharp vorticity gradients in two-dimensional turbulence and the energy spectrum

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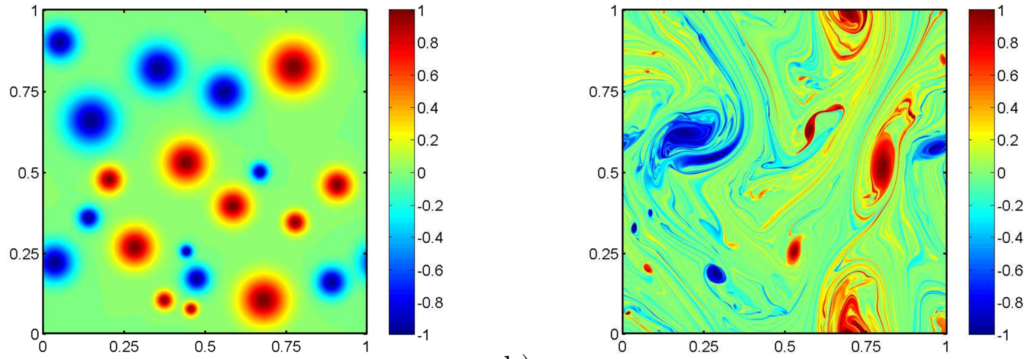
We present investigations of the relation between vortical structures and the energy spectrum in two-dimensional (2D) turbulent flows with particular attention to the formation of sharp vorticity gradients and their influence on the spectrum in the large wavenumber regime.

There are two kinds of well-known 2D turbulent spectra. The first one, derived by Kraichnan [1], corresponds to an enstrophy cascade toward the small-scale regime, where viscous dissipation becomes essential. The Kraichnan spectrum has, up to a logarithmic factor, a power law dependence in the inertial range: $E(k) \sim k^{-3}$. (Recall that 2D turbulence also possess an inverse energy cascade toward large scales leading to the Kolmogorov dependence $E(k) \sim k^{-5/3}$ [1]). The other spectrum, obtained by Saffman [2], has a different power dependence: $E(k) \sim k^{-4}$. This is ascribed to sharp vorticity gradients forming in decaying 2D turbulence at high-Reynolds numbers. Under the assumption of isotropy and a dilute distribution of the localized regions of sharp gradients Saffman constructed the energy spectrum at large k as a superposition of the spectra from the gradients.

We present qualitative arguments for the formation of sharp vorticity gradients in in 2D flows for smooth initial conditions. The main idea is to apply an analog of the so-called vortex line representation (VLR) introduced for three-dimensional vortical flows [3]. This representation is based on a mixed Lagrangian-Eulerian description and connected with movable vortex lines. The VLR is a mapping to a curvilinear system of coordinates and turns out to be compressible, which appear to be a mechanism for enhancement of vorticity and may lead to formation of singularities. For 2D flows the vorticity is a Lagrangian invariant quantity and cannot be locally enhanced. However, the so-called di-vorticity, $\mathbf{B} = \nabla \times \omega \hat{z}$, represents a frozen-in field, i.e., it satisfies an equation similar to the vorticity equation for 3D Euler flows. Therefore, we can apply the VLR to the di-vorticity field. Thus, the di-vorticity lines are compressed leading to a local enhancement of the di-vorticity and hence of the vorticity gradient, which may grow very large, but cannot become infinite in finite time.

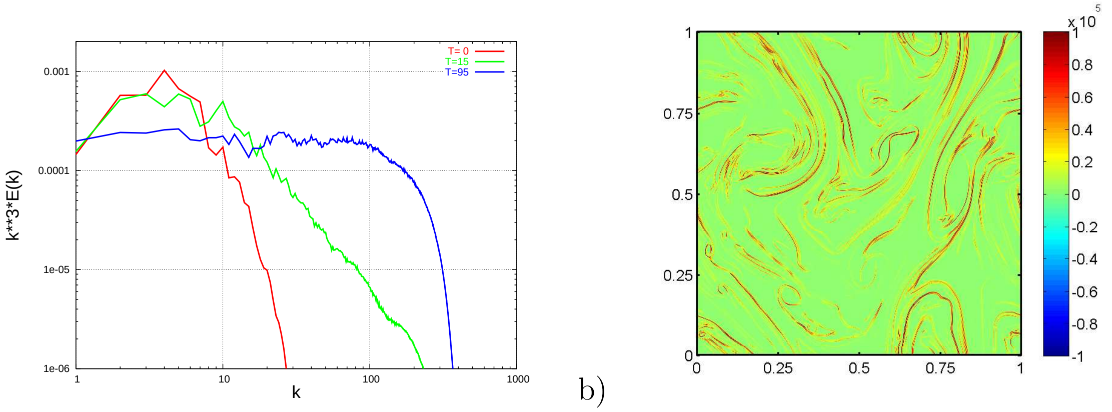
Considering the effect of these sharp vorticity gradients on the energy spectrum we follow Saffman [2]. Using the stationary phase method we demonstrate that the contribution from one discontinuity is very anisotropic: it has a sharp angular peak along the direction perpendicular to the discontinuity. In the peak the energy spectrum falls-off like k^{-3}. Assuming a dilute distribution of the sharp gradients and averaging over all angles in the case of isotropic turbulence the spectrum becomes k^{-4}. However, in the case of anisotropy, where the stripes of sharp vorticity gradients are along almost straight lines, the angle averaging results in a spectrum: k^{-3} [4].

To support the arguments above and reveal the connection between the formation of the sharp vorticity gradients and the tail of the energy spectrum, we have performed a numerical study of the evolution of decaying 2D turbulence [4]. Figure 1 shows the evolution



a) b)

Figure 1: a) Initial vorticity field. b) Vorticity field at time 100 corresponding to ≈ 8 vortex turnover times.



a) b)

Figure 2: a) Compensated energy spectrum $k^3E(k)$ corresponding to the vorticity field in Fig. 1. b) The squared length of the di-vorticity vector $|\mathbf{B}|^2$ at time 100.

of the vorticity from an initial distribution of randomly placed vortices all with amplitude $\omega_0 = 1$, Gaussian profiles and different sizes. At time $= 100$ the vorticity field has the typical structure for 2D turbulence; it consist of large scale structures (vortices) with concentrated vorticity and strongly filamented structures between the vortices. Corresponding to the vorticity field we show the instantaneous one-dimensional energy spectrum $E(k)$ in Fig. 2. For $t = 0$ $E(k)$ is the spectrum of superimposed Gaussian vortices, and at $t = 95$ a k^{-3} spectrum has developed at high wave numbers. Thus, with reference to the discussion of the Saffman spectra above this corresponds to the spectrum expected in the anisotropic regime where the stripes of vorticity gradients are near straight lines. In Fig. 2b we observe stripes of strongly amplified - up to thousand times - di-vorticity field, and indeed the stripes are close to straight lines.

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