Vortex dynamics of wakes

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Several problems related to the dynamics of vortex patterns as observed in wake flows are addressed. These include: The universal Strouhal-Reynolds number relation. The Hamiltonian dynamics of point vortices in a periodic strip, both the classical two-vortices-in-a-strip problem, which gives the structure and self-induced velocity of the traditional vortex street, and the three-vortices-in-a-strip problem, which is argued to be relevant to the wake behind an oscillating body. The bifurcation diagram for wake structure found experimentally by Williamson and Roshko is addressed theoretically.

1 Introduction

Vortex street wakes are ubiquitous. We can create them in the laboratory and we observe them in Nature. We see them in planetary atmospheres. Thus, in recent years spectacular vortex street wakes at very high Reynolds number have been observed “behind” certain islands in satellite images (cf. Fig. 1). We realize their profound effect from instances such as the collapse of the Tacoma Narrows Bridge on November 7, 1940.

While the phenomenon of vortex streets had been observed qualitatively for many years, it was not until the seminal work of T. von Kármán in 1911-12 [13, 14, 15] that the first theory of these structures was produced. So important was this contribution of von Kármán that the Hungarian postage stamp commemorating him (issued in 1992) shows his portrait on a background of the streamline pattern (in the co-translating frame) of the particular staggered vortex street that he identified as being not linearly unstable (see Fig.2). I shall return to von Kármán’s contributions in Section 4. Let me first mention another very important result that has emerged, mostly from experiment, namely the well known relation between the *Strouhal number* for vortex shedding into the wake and the *Reynolds number* of the wake-generating flow (see Fig.3). Let us first ask: How might one think about such a relation theoretically?

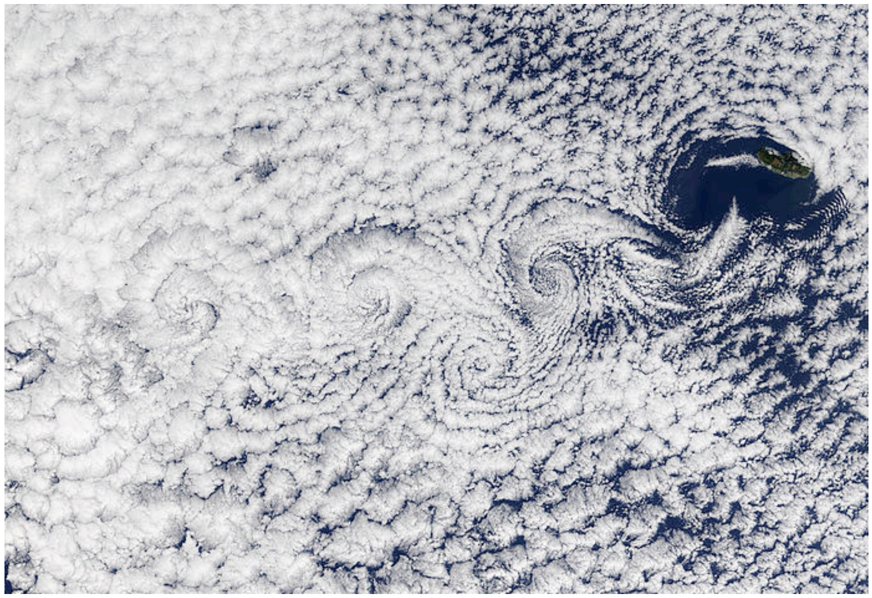

Fig. 1. NASA satellite image of 26 April 2002 showing a well-developed vortex street behind Madeira island.

2 The Strouhal-Reynolds number relation

Empirically one finds that the Strouhal number, which is the non-dimensional shedding frequency, depends on the Reynolds number of the wake-producing flow as

$$St = 0.2175 - \frac{5.1064}{Re}, \tag{1}$$

for the "laminar" regime (regime I in Fig.3; up to $Re \approx 200$), and by

$$St = 0.212 - \frac{2.7}{Re}, \tag{2}$$

for large values of Re, say 400 and higher. The latter fit includes the famous limiting value (0.212) of the Strouhal number at high Reynolds number. Of course in this second regime (regime II in Fig.3) the flow does not just respond with one frequency but the Strouhal number corresponds to the frequency with most of the energy.

There is a "transition" regime (regime III in Fig.3) where the curve seems to break. This regime is related to three-dimensional vortex motion in the wake. I shall not have anything to say about this regime in this paper.

We have approached this problem in the following way: First, we recall that the Navier-Stokes equations only give us that St is a function of Re, i.e., that there must exist some functional relation $St = f(Re)$ where f is (somehow) to be determined from the equations of motion and the shape of the body. Second, we assume – based on an analogy to the phenomenology of phase transitions or general ideas from bifurcation theory – that close to the bifurcation the Strouhal number depends as a power law on the deviation of $1/Re$ from its “critical” value at the bifurcation, i.e., we should expect for $Re \approx Re_{crit}$ that

$$St = A \left(\frac{1}{Re_{crit}} - \frac{1}{Re} \right)^{\alpha}, \tag{3}$$

where Re_{crit} is the bifurcation Reynolds number (which is non-universal), A is a non-universal coefficient, but α is a universal exponent. Experiment further suggests that $\alpha = 1$ which points to a mean-field theory of the phenomenon.

Fig. 2. Hungarian postage stamp memorializing von Kármán. In the background the streamline pattern for a staggered point vortex street.

So, what equation should one try to apply a “mean-field analysis” to? We [24] thought the two-dimensional vorticity equation,

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{V} \cdot \nabla \zeta = \nu \nabla^2 \zeta, \tag{4}$$

was a natural candidate. In the paper just cited we estimate the terms in this equation as follows:

$$\frac{\partial \zeta}{\partial t} \approx f \Delta\zeta, \quad \boldsymbol{V} \cdot \nabla \zeta \approx U \frac{\Delta\zeta}{d}, \quad \nu \nabla^2 \zeta \approx \nu \frac{\Delta\zeta}{d^2}. \tag{5}$$

Here f is the shedding frequency, which sets a natural time scale for the flow, U is the free stream velocity and d the diameter of the cylinder. The quantity

$\Delta\zeta$ gives the scale of vorticity fluctuations in the emerging wake. There are points of large vorticity, primarily in the vortices that are forming to make up the vortex street, and there are points of smaller vorticity in sheets and other "background" flow structures that will ultimately be swept up into the vortices.

In our paper [24] we argue, based on careful examination of the vortex wake formation process in a numerical simulation [26], that the viscous term acts exclusively to spread out and impede vortex formation, i.e., in the vorticity balance in the near wake this term should be viewed as a sink when writing the vorticity balance. We also argue that part of the advective term on the right hand side acts to assemble the vortices (the rest simply advects the vorticity downstream). This is a source term for vortex generation and should enter the vorticity balance with a positive sign. Based on this kind of order of magnitude estimates and physical reasoning to determine the signs of the various contributions, we recast the vorticity equation in the following form (in terms of orders of magnitude with signs):

$$f\,\Delta\zeta = k_a U \frac{\Delta\zeta}{d} - k_d \nu \frac{\Delta\zeta}{d^2}, \tag{6}$$

where k_a and k_d are two dimensionless parameters that require a more comprehensive analysis to determine. It is easily seen that this relation, after cancellation of $\Delta\zeta$ from all terms and multiplication by d/U, is precisely of the form of the empirical Strouhal-Reynolds number relation.

There are a number of questions one can ask of this simple "derivation", e.g., whether it is satisfactory that $\Delta\zeta$ cancels out of all the terms[1]. In particular, the crude estimate for the advective term may seem dubious. We will not enter into a discussion of these issues here (for more detail see the paper cited) but simply state the suggestion that the correct and fully rigorous approach to this problem requires finding a similarity solution of the vorticity equation that, somehow, applies to vortex shedding. We leave this as a challenge to the reader!

3 Hamiltonian dynamics of point vortex dynamics

The point vortex model originated with Helmholtz's seminal 1858 paper on vortex dynamics [17]. The most elegant statement arises if one concatenates the x- and y-coordinates of the vortices into complex positions $z_\alpha = x_\alpha + \mathrm{i} y_\alpha$, $\alpha = 1, 2, ..., N$. Then the equations of motion take the form

$$\dot{z}_\alpha^* = \frac{1}{2\pi \mathrm{i}} \sideset{}{'}\sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{7}$$

[1] I am indebted to T. Bohr for raising this point.

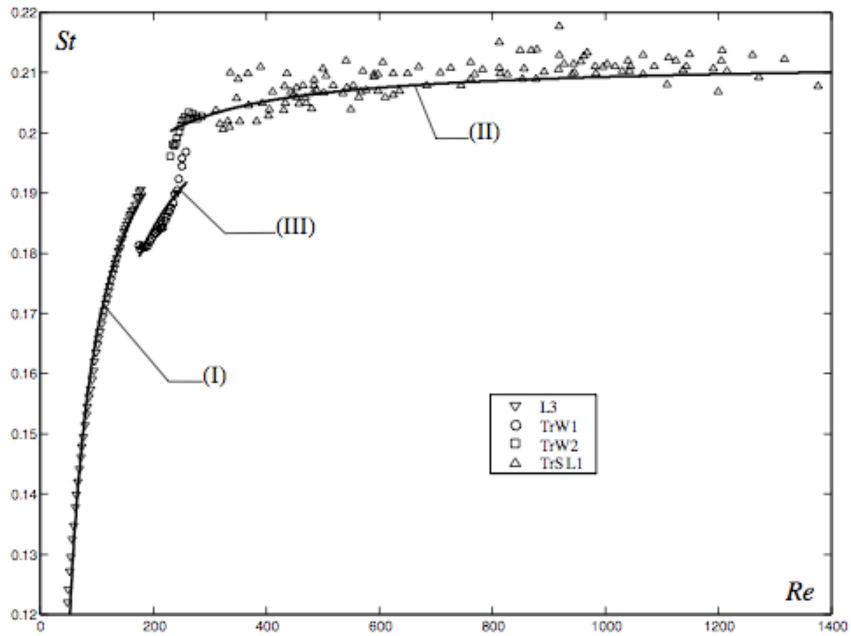

Fig. 3. The empirical Strouhal-Reynolds number relation for flow behind a cylinder. Different symbols have been used for different vortex shedding regimes abbreviated L3, TrW1, etc. The overall division into three regimes, I, II and III, is described in the text.

Here the Γ_β are the circulations of the vortices, invariant in time by Helmholtz's theory – even better, maybe, by Kelvin's circulation theorem – the asterisk on the left hand side denotes complex conjugation, the dot differentiation with respect to time, and the prime on the summation symbol reminds us to skip the singular term $\beta = \alpha$.

Helmholtz gave the solution of the two-vortex problem, where he showed that two vortices would have orbits on concentric circles, which in the special case of a vortex pair degenerate to translation along parallel lines.

A major formal development of the theory was provided by Kirchhoff [18], who in his lectures on theoretical physics, published in several editions starting in 1876, showed that the point vortex equations could be recast in Hamilton's canonical form:

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \tag{8}$$

where the Hamiltonian, H, is

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|. \tag{9}$$

Again we exclude the singular terms $\alpha = \beta$ and remind ourselves to do so by placing a prime on the summation. A complete correspondence with Hamilton's form of the equations of motion is obtained by choosing the generalized coordinates to be $q_\alpha = x_\alpha$ and the generalized momenta to be $p_\alpha = \Gamma_\alpha y_\alpha$. This also shows that for vortices *phase space is configuration space*, a feature that has profound consequences for both the statistical physics of point vortices and for the phenomenon of chaotic advection [2]. Many of these aspects were covered by other speakers at the symposium.

The Hamiltonian nature of the point vortex equations immediately leads to important insights about the availability of integrals of the motion and, in turn, about integrability of the N-vortex problem. Thus, the invariance of H to translation and rotation of coordinates, and its independence of time, leads to the integrals X, Y and I given by

$$X + \mathrm{i}Y = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha, \qquad I = \sum_{\alpha=1}^{N} \Gamma_\alpha |z_\alpha|^2, \tag{10}$$

and, of course, H itself. The quantities X and Y are the two components of the *linear impulse*. The quantity I is the *angular impulse*.

Pursuing the formalism of classical dynamics a bit further, we introduce the *Poisson bracket*

$$[f, g] = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right). \tag{11}$$

The *fundamental brackets* may be written

$$[z_\alpha, z_\beta] = 0, \qquad [z_\alpha, z_\beta^*] = -\frac{2\mathrm{i}}{\Gamma_\alpha} \delta_{\alpha\beta}. \tag{12}$$

We now obtain the key results

$$[X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha, \qquad [X.I] = 2Y, \qquad [Y, I] = -2X, \tag{13}$$

from which the very important result

$$[X^2 + Y^2, I] = 2X[X, I] + 2Y[Y, I] = 0 \tag{14}$$

follows. These results show (a) that no new integrals arise by taking Poisson bracket of the known integrals, and (b) that the problem always has three independent integrals *in involution*, namely $X^2 + Y^2$, I and H.

Poincaré realized as much in his lectures of 1891-92 [23] and concluded (from what we today call Liouville's theorem) that the three-vortex problem on the unbounded plane is always integrable. Apparently this was not of sufficient interest to him and he never returned to the problem. The general formalism given above was pursued by the Italian E. Laura in a number of papers early in the 20th century [20] but then lay dormant for decades.

Actually some 15 years before Poincaré's work the three-vortex problem had been completely solved by a young Swiss mathematician W. Gröbli whose 1877 thesis [12] was for some reason overlooked[2] for about a century. Even the revival of Gröbli's work in an important paper [31] by J. L. Synge for the inaugural issue of the *Canadian Journal of Mathematics* in 1949, an issue that contained a seminal paper in general relativity by Einstein and Infeld, failed to introduce the solution of this three-body problem into the mainstream of fluid mechanics. For a review of this history see [5].

It turns out that there is a bit of a "hole" in the treatments of Gröbli and the later work by Synge, Novikov and the author [31, 22, 1] concerning the special case $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. While being covered in principle by the general analysis, it admits of a much more complete discussion. This was provided by Rott [28] and the author [3]. In essence what our treatment of the problem shows is that the relative separation of two of the vortices, say vortices 1 and 2, i.e., $Z = z_1 - z_2$, evolves as if it were the position of a fictitious passive particle in the field of three *fixed* vortices. The strengths and locations of the three fixed vortices are given by the strengths of the original three vortices and the linear impulse of the original three-vortex system. Thus, if the original three vortices have strengths Γ_1, Γ_2, Γ_3, the three fixed vortices in the advection problem have strengths Γ_1^{-1}, Γ_2^{-1}, Γ_3^{-1}. (All that matters is really the proportion of the vortex strengths – the absolute value can be absorbed in a rescaling of space and time.)

This reduction of the problem – from three points corresponding to the three original vortices, to one point corresponding to an advected particle – is somewhat akin to what happens in the Kepler problem of celestial mechanics, where the motion of two interacting mass points is decomposed into a trivial center-of-mass motion and a relative motion. It leads to the following scenario: There is the *physical plane* where the motion of the three vortices takes place, i.e., the vortex positions z_1, z_2, z_3 "live" in this plane. There is a *phase plane* where the advection of the fictitious particle takes place, i.e., Z evolves in this plane.

For three vortices on the infinite plane the advection problem in the phase plane is relatively simple. There are four distinct regimes of motion. Three of these arise in the obvious way through two of the vortices being closer to one another than to the third vortex, and hence moving as if in a "bound state". The fourth regime corresponds to truly "collective states" where all three vortices interact continuously.

[2] This happened in spite of references to it in Kirchhoff's lectures (2nd ed.) [18] and in Lamb's well known text [19].

4 Point vortex modeling of wakes

It turns out that the solution method for three vortices on the infinite plane can be extended to the problem of three vortices in a domain with periodic boundary conditions as was first shown by Aref & Stremler [6, 30]. In the case of vortices in a periodic strip, which is the case that is most immediately applicable to vortex wakes, one has to stipulate that $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ just as on the infinite plane. (In the case of vortices in a periodic parallelogram the periodicity of the flow assures that the sum of the "base" vortices in the basic parallelogram is zero.) The equations of motion for vortices in a periodic strip of width L are

$$\dot{z}_\alpha^* = \frac{1}{2L\mathrm{i}} \sum_{\beta=1}^{N}{}' \Gamma_\beta \cot\left[\frac{\pi}{L}(z_\alpha - z_\beta)\right]. \tag{15}$$

These equations appear first to have been written down in 1928 by Friedmann & Poloubarinova [11]. See also [27].

With the wisdom of hindsight one may say that von Kármán's theory of the structure of the vortex street follows from (15) with $N = 2$ and $\Gamma_1 = -\Gamma_2 = \Gamma$ and, thanks to later work by Domm [10], his theory of the stability of vortex streets follows almost entirely, although not quite, from (15) with $N = 4$ and $\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = \Gamma$. (Probably the most accessible account of von Kármán's theory for the modern reader is the exposition in [19].)

In brief, von Kármán's theory of the vortex street shows, first, that the only two-vortex-per-strip configurations to propagate downstream are the symmetric and the staggered configuration. From the two-vortex version of (15) one easily deduces that a $\pm\Gamma$ pair in a periodic strip propagates with velocity

$$U - \mathrm{i}V = \frac{\Gamma}{2L\mathrm{i}} \cot\left[\frac{\pi}{L}(z_+ - z_-)\right]. \tag{16}$$

For the velocity to be real, i.e., in order to have $V = 0$ in (16), the cotangent must be pure imaginary. This implies $\Re(z_+ - z_-) = 0$ or $\Re(z_+ - z_-) = L/2$. The first possibility corresponds to symmetric vortex streets, the second to staggered vortex streets.

Von Kármán next considered the stability of these two types of configurations. He did, in essence, two stability calculations, in both cases working with infinite rows of vortices. In the first he simply perturbed one vortex keeping all the others fixed. This calculation showed that the symmetric configuration was always linearly unstable and the staggered configuration was linearly unstable unless the ratio of $b = \Im(z_+ - z_-)$ and the inter-vortex distance in each row, h, has a certain value. (To avoid confusion we use a new symbol, h, for the distance between vortices in either row because for, say, four-vortices-in-a-strip the period of the strip, L, is related to the inter-vortex distance by $L = 2h$, whereas $L = h$ for the two-vortices-per-strip case.) In fact, in his first attempt von Kármán produced the erroneous result $\sinh(\pi b/h) = \sqrt{2}$. (The reason for this "error" is that when perturbing just one vortex one is adding linear momentum and kinetic energy to the system being perturbed. The appropriate criterion arises from perturbations that do not add linear momentum or energy.) The correct result, which von Kármán quickly produced as well, and which is today known as his famous stability criterion for vortex streets is

$$\sinh \frac{\pi b}{h} = 1. \tag{17}$$

The main thrust of our work on more complicated vortex wakes – we have used the term "exotic" – is to apply the solution for three-vortices-in-a-strip that we have found to model these in the same spirit that von Kármán modeled steady vortex streets by the two-vortices-in-a-strip solutions. An example of an "exotic" wake with three vortices shed per cycle is show in Fig.4. It is a tenet of vortex wake dynamics, apparently true but difficult to prove, that the total circulation of all vortices shed during one cycle is zero. This applies also to such cases as a cylinder oscillating normally to an oncoming uniform flow.

A recent paper by Ponta, Stremler and the author [4] gives a rather thorough exposition of our ideas so we shall be content with a brief summary here.

In the extension of the solution for three vortices with sum of circulations equal to zero to periodic boundary conditions [6, 30] one finds, once again, that the problem can be "reduced" to an advection problem for the relative position of two of the vortices, say again $Z = z_1 - z_2$. This time, however, the advecting system of vortices consists of three rows of advecting vortices, not just three vortices. The vortices in each of the three rows are identical, and their circulations are, respectively, Γ_1^{-1}, Γ_2^{-1}, Γ_3^{-1} (modulo rescaling of the time). Indeed, the position of the "base vortex" in each row is given exactly as in the unbounded plane case in terms of the linear impulse of the system and the circulations. It turns out that if the ratio of the circulations is rational (and because the sum is zero, if the ratio of two circulations is rational, the ratio of any two circulations is rational), the three rows of advecting vortices fit into a periodic strip with a width that is a multiple of the period L of the strip in the physical plane. If the ratios are irrational, the three rows of advecting vortices have no common period and we are faced with advection by an infinite system of stationary vortices.

Again an advection problem in the phase plane arises but this time with a more complicated structure of the various *regimes of motion* than in the unbounded plane case. There are, in general, many more regimes for Z to wander through and thus many more regimes for the vortex motion itself. (To find z_1, z_2 and z_3 from Z requires an additional quadrature.) This provides the first qualitative conclusion: Vortex wakes with three (and, thus, presumably with more than three) vortices shed per cycle provide a considerably richer variety of wake patterns than the classical vortex street wakes (and we include under this rubric both the von Kármán street and its oblique "cousins" found subsequently by Dolaptschiew and Maue, cf.[21]). Furthermore, so far as we can tell, the richness in the dynamical structure of the three-vortices-in-a-strip

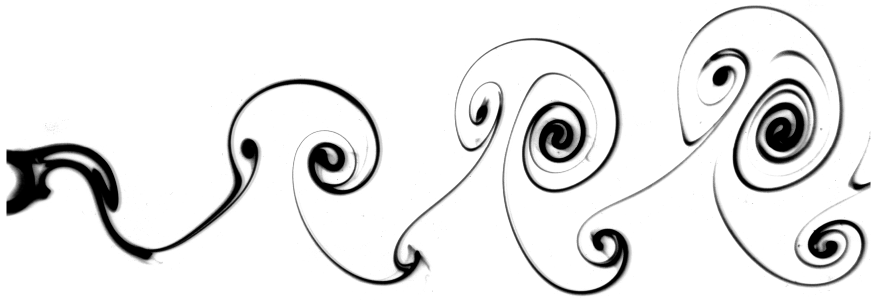

Fig. 4. "Exotic" vortex street wake behind an oscillating cylinder (courtesy of C. H. K. Williamson).

solution is only partially reflected in the known experimental results. The phase plane diagrams reveal, for example, a multitude of relative equilibria with three vortices per period, none of which have been observed. Somewhat surprisingly, these can be determined analytically [29]. They are all linearly unstable (since they correspond to saddle points in the phase plane diagram), which may explain why they do not occur (or do not seem to occur) even as transients in experimental images of vortex wakes. However, a thorough analysis of such images has yet to be undertaken, and we have only recently understood what to look for.

We also believe that it is possible to generalize the *Kármán drag law*, that was derived for the ordinary vortex street [15], to a certain class of more complicated vortex wakes. Work is in progress on this topic. Experimental results suggest considerable richness in the structure of the drag force versus the frequency of oscillation of the cylinder. It would be interesting to produce such results using the simple wake models considered here.

5 Bifurcation diagram for vortex wakes

A persistent problem in relating the analytical solutions to real wake experiments is the difference in control that one has over initial vorticity distributions in experiment versus theory. In the theory the locations and circulations of a set of vortices is given as an initial condition. In experiment these data arise through a complex process of boundary layer instability, vortex sheet roll-up, and vortex formation. We often refer to this process simply as "vortex shedding", although the wake vortices are typically not "shed" ready-made for assembly into a wake. The process is considerably more intricate and involves several stages.

The controls that the experimenter has are such things as amplitude and frequency of oscillation of the wake-producing cylinder, the shape of the cylinder, and the velocity of the oncoming free stream. How to "map" these controls onto the nature of the resulting wake is at present still something of an art. The most reliable guide we have in this realm is the bifurcation diagram determined experimentally by Williamson & Roshko [32] for wakes produced by a cylinder oscillating normally to an oncoming uniform free stream. A number of different wake formation modes, labeled by “S” for singlet, “P” for pair, and various combinations thereof (e.g., the wake in Fig.4) would be “S+P”) were identified and delineated in a plot that has as its abscissa the wavelength of the oscillatory motion of the cylinder and as its ordinate the amplitude of that same oscillation (both coordinates non-dimensionalized by the cylinder diameter).

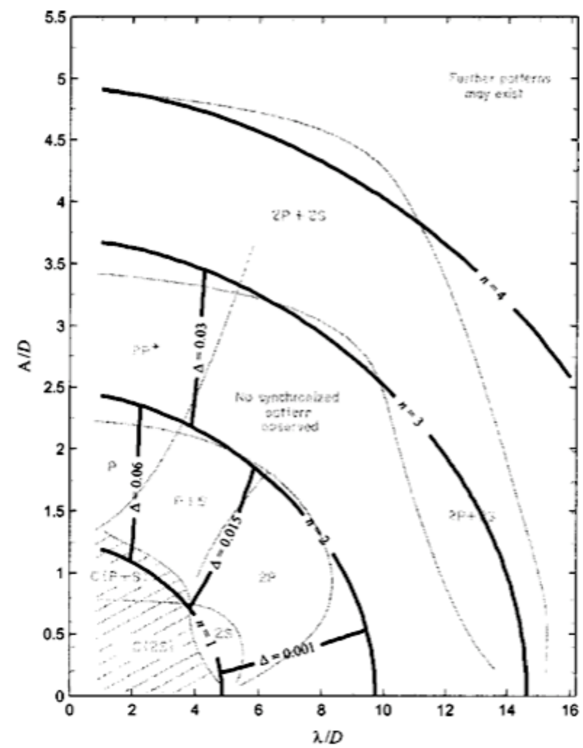

Fig. 5. Theoretical contours superimposed on the Williamson-Roshko bifurcation diagram (background, blurred). The radial contours, Eq.(18), have no adjustable parameters.

We have recently tried to provide some theoretical ideas to “rationalize” the structure of this diagram. A crude approximation suggests that the dividing line between the various regimes in the Williamson-Roshko diagram are radial and circumferential. Also, in general, there are more vortices shed per cycle as one goes farther out radially from the origin in the diagram. In our paper [25] Ponta and I consider the undulatory motion of the cylinder as a sequence of rectilinear "strokes" interrupted by sharp turns. During any "stroke" the cylinder sheds much as it would in an oncoming steady uniform stream. There is one complication: Since the experiment is conducted by giving a constant streamwise velocity to the cylinder, the speed along the actual path varies. Hence the Reynolds number varies and so, because of the Strouhal-Reynolds number relation, the shedding frequency varies. Apart from this effect – which is akin to driving an oscillator with a slightly varying forcing frequency – the length of the rectilinear "stroke" determines how many vortices are shed. Thus, we have both an effect of the amplitude and of the wavelength of the oscillation on how many vortices are shed per "stroke". Since the number of vortices that are recognized in the wake patterns in terms of pairs and singlets is an integer, there is a "quantization" of the resulting wake as a function of the continuously variable control parameters (i.e., wavelength and amplitude of oscillation). This quantization can be expressed by a formula

$$\frac{\lambda}{D} St\ E\left(-(2\pi A/\lambda)^2\right) = n\frac{\pi}{2}. \tag{18}$$

Here λ and A are, respectively, the wavelength and amplitude of oscillation of the cylinder, St is the Strouhal number corresponding to the Reynolds number for the free stream according to the $St - Re$ relation, and n is an integer. The function E is the complete elliptic integral of the second kind. See [25] for a derivation. The curves (18) correspond to the radial delimiting lines in the Williamson-Roshko bifurcation diagram. When they are plotted in that diagram, the correspondence is surprisingly good (and we note that there are no adjustable parameters).

It is more difficult to produce a convincing theory for the radial lines that divide shedding regimes in the Williamson-Roshko diagram. We believe these delineations are related to a threshold tolerance of the vortex shedding process to variations in Reynolds number (and, hence, in the corresponding Strouhal number) during the oscillatory motion of the cylinder. As we have already said, the instantaneous Reynolds number for flow about the cylinder varies in the course of its motion because the streamwise velocity is held constant in the experiment. Hence the speed along the undulatory path must vary and it is this speed relative to the fluid that sets the shedding. However, what such a threshold might be is difficult to tell without a more detailed quantitative model of the shedding process itself, something that we do not currently possess. The radial lines in Fig.5 are drawn by choosing values for such a threshold to give a best fit to the experimentally observed lines. Thus, what one can say, at best, is that the qualitative explanation may have some validity. A deeper quantitative understanding awaits.

6 Concluding remarks

It seems fair to say that vortex dynamics gives conclusions about wake structure and stability that are difficult to obtain in any other way. Using the wake vortices as the main degrees of freedom in the theory gives an entirely different perspective than other approaches based, for example, on linear or even non-linear stability considerations.

In the “vortex representation” the problem of two vortices in a periodic strip gives the structure and translation velocity of the von Kármán vortex street configurations. Four vortices in a periodic strip give the stability criterion and show, in particular, that even the staggered configuration singled out by von Kármán's linearized stability analysis is not stable when second order perturbations are taken into account.

The problem of three vortices per strip gives an access point to the lowest order modes of vortex streets observed behind an oscillating cylinder. The theory of of three-vortices-per-strip currently appears much richer than observations of wakes behind oscillating cylinders in the sense that there are many regimes of motion suggested by the analysis that do not seem to have experimental counterparts. This can simply be the result of an incomplete analysis of the current experimental results or it can be the result of the restricted access to the full parameter space of the problem that can be achieved when the vortices must be produced through shedding from an oscillating cylinder. We note that there are a multitude of stationary patterns (relative equilibria) but that they are all linearly unstable. This would imply that they must be sought in an analysis of slow transients in the wake evolution, not necessarily as immediately produceable steady states.

Based on what happens on the unbounded plane, one would assume that wakes with four (or more) vortices shed per cycle lead to configurations with chaotic motion and that no patterns would be expected. Both the premise and the conclusion in this statement require further work. We may add that we now understand the problem of three vortices in a periodic strip to be “maximally chaotic” in the sense that the advection it produces can be a *pseudo-Anosov mapping* in a certain region of the flow. We refer the reader to the recent work on *topological chaos* by Boyland, Stremler and the author [8, 9] for an exposition and explanation of these statements. The onset of topological chaos for advection by three vortices strongly supports the contention above that wakes with four or more vortices shed per cycle will not show discernible patterns.

Finally, it seems clear that in spite of the voluminous literature on vortex wakes, there are still many open problems, even for completely 2D flow. The new “exotic” wakes revealed by oscillating the vortex-producing body have opened up a Pandora's box of possibilities that we are only beginning to grasp theoretically.

My research in this area has been done in close collaboration with F. L. Ponta and M. A. Stremler whose insights are reflected in the above (all errors and misstatements, of course, being my responsibility). Fruitful discussion with C. H. K. Williamson are also gratefully acknowledged. I thank the organizers, in particular Mikhail A. Sokolovskiy and Olga I. Yakovenko, for their care and hospitality. This work is supported by a Niels Bohr Visiting Professorship at the Technical University of Denmark funded by the Danish National Research Foundation.

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