A generalization of the Helmholtz-Kirchhoff model

C. Eugene Wayne[1,3] Raymond Nagem[2] Guido Sandri[2]
David Uminsky[1,3]

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Abstract

We give a convergent expansion of solutions of the two-dimensional, incompressible Navier-Stokes equations which generalizes the Helmholtz-Kirchhoff point vortex model to systematically include the effects of both viscosity and finite core size. The evolution of each vortex is represented by a system of coupled *ordinary* differential equations for the location of its center, and for the coefficients in the expansion of the vortex with respect to a basis of Hermite functions.

We study the evolution of the vorticity of a tw0-dimensional, incompressible fluid in the plane:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega \ . \tag{1}$$

If the vorticity is integrable, we can recover the velocity field via the Biot-Savart law, so that the term $\mathbf{u}$ in the equation can be regarded as a linear but nonlocal function of the vorticity ω.

We then expand the vorticity as a sum of N-components:

$$\omega(x,t) = \sum_{j=1}^{N} \omega^j(x - x^j(t); t) \tag{2}$$

with a similar decomposition of the velocity field. (i.e. we expand $\mathbf{u}$ as a sum of components $\mathbf{u}^j$ where $\mathbf{u}^j$ is the velocity field associated to the vorticity ω^j.) Here, $x^j(t)$ is the center of the j^{th}-vortex - we define its evolution below.

Note that this decomposition is not unique - indeed, even the number N of components is not uniquely defined, though the expansion is most useful when the flow consists of distinct, relatively isolated regions of non-zero vorticity. However, the expansion below is

[1]Dept. of Mathematics and Statistics and Center for Biodynamics, Boston University
[2]Department of Aerospace and Mechanical Engineering, Boston University
[3]Supported in part by NSF grant DMS-0405724

well defined, no matter how the vorticity is decomposed, provided only that each component ω^i has non-zero total vorticity.

Each component of the vorticity is now expanded as a series with respect to the basis functions $\phi_{j_1,j_2} = D_{x_1}^{j_1} D_{x_2}^{j_2} \phi_{0,0}(x_1, x_2)$, where $\phi_{0,0}$ is the Gaussian vorticity distribution that defines the Oseen vortex solution of the two-dimensional Navier-Stokes equation, i.e.

$$\phi_{0,0}(x_1, x_2) = \frac{1}{\pi \lambda^2} e^{-|x|^2/\lambda^2}$$

where $\lambda^2 = \lambda_0^2 + 4\nu t$. The parameter λ_0 describes the initial width of the j^{th} vortex core. In general one can take a different λ_0 for each vortex but for simplicity we assume that all vortices have the same value of λ_0 here. The expansion of the j^{th} vortex takes the form:

$$\omega^j(z,t) = \sum_{k_1,k_2=1}^{\infty} M^j[k_1, k_2; t] \phi_{k_1,k_2}(z,t) \ . \tag{3}$$

In order to fix the evolution of the centers of the vortices we require that the first moment of each vorticity region must vanish at every time $t > 0$, i.e. we require that

$$\int_{\mathbb{R}^2} (x - x^j(t)) \omega^j(x - x^j(t), t) dx = 0 \quad \text{for all} \quad t > 0, \ j = 1, \dots N \ . \tag{4}$$

The motivation for this definition is recent theoretical work of Gallay and Wayne on the stability of Oseen's vortex which shows that the rate of approach to the vortex is increased if the first moments of the vorticity distribution are zero.

We next derive an infinite set of coupled ordinary differential equations for the evolution of the coefficients $M^j[k_1, k_2; t]$ and $x^j(t)$ and prove the following results about these equations:

- We derive conditions on the initial vorticity distribution that are sufficient to guarantee that the expansions in (3) are convergent for all time $t > 0$.
- The ordinary differential equations for the coefficients and vortex centers contain only quadratic terms and we derive a simple, expression for these coefficients in terms of derivatives of a single, explicit kernel function.
- We show that in the limit in which the vortices tend to point vortices and the viscosity tends to zero, the system of equations for the coefficients and centers of the vortices converges to the classical Helmholtz-Kirchhoff equations for a system of inviscid point vortices. Thus, our expansion gives a systematic way to incorporate the effects of viscosity and finite core size into the Helmholtz-Kirchoff model.
- We show that in the case in which we have a pair of vortices that can be approximated by Oseen vortices, (or equivalently, when we truncate our expansion of each vortex at lowest order) our model equations are explicitly solvable and they show that the vortex centers move either along arcs of circles or along straight lines.