An example of finite-time singularities in the 3d Euler equations

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1. Introduction. The 3d Euler equations describe an ideal, incompressible fluid:

$$\partial_t v + v \cdot \nabla v = -\nabla \pi, \qquad \text{div } v = 0 \quad \text{in } \mathbb{R}^3 \times (0, T], \tag{1}$$

where $v(x,t) = (v_1, v_2, v_3)$ denotes the velocity field, and π the pressure. Given C^∞ divergence-free initial data $v(x,0) = v_0$, $v_0 \in L^2(\mathbb{R}^3)$, it is an open question as to whether $v(x,t)$ is regular at $T = +\infty$ (cf. [C]).

In this paper, we shall construct a special class of weak solutions to the Euler equations which blow up at a finite time $t = T$. Our key idea is to search for

$$\begin{cases} v(x,t) = v_s + u(x,t), \quad \pi(x,t) = \pi_s + p(x,t), \\ \text{div } v_s = \text{div } u = 0, \quad v_0 = v_s^0 + u_0 \in H^m(\mathbb{R}^3), \quad m \geq 3. \end{cases} \tag{2}$$

Above (v_s, π_s) is the self-similarity transformation for the Euler equations:

$$v_s(x,t) = V(y)(T-t)^{-\alpha}, \quad \pi_s(x,t) = \Pi(y)(T-t)^{-2\alpha}, \quad y = x(T-t)^{-\beta}, \tag{3}$$

where $\beta \in [2/5, 1], \alpha + \beta = 1,\ \alpha, \beta > 0$ and $T < +\infty$. Substituting (3) into (1), the profile velocity $V(y)$ satisfies the system

$$\alpha V + \beta y \cdot \nabla V + V \cdot \nabla V + \nabla \Pi = 0, \qquad \text{div } V = 0. \tag{4}$$

For $\alpha = \beta = 1/2$, (4) is the limiting case of Leray's self-similar Navier-Stokes equations. If $V(y) \not\equiv 0$ is found, then v_s given by (3) blows up at $t = T$, consistent with the criterion [CFM]. Hence our task is to find $v(x,t) = v_s + u$ defined on $\mathbb{R}^3 \times (0,T)$, with $u(x,t)$ being in an appropriate function space.

2. Existence of self-similar solutions v_s in an exterior domain. Let us begin with summarising a previous result:

Proposition A. *(cf. Theorem 3C [H]). Let $\alpha = \beta = 1/2$ and $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$. Denote by σ and τ the unit normal and the unit tangent to $\partial\Omega$, respectively. Suppose that $f \in W^{2,4}(\partial\Omega; \mathbb{R})$, $g \in W^{3,4}(\partial\Omega; \mathbb{R})$, and that these functions are small in appropriate norms. Then there exists a unique, stable solution $(V, \Pi) \in C^1(\Omega; \mathbb{R}^3 \times \mathbb{R})$ to the system (4), satisfying the prescribed data*

$$\begin{cases} \sigma \cdot V|_{\partial\Omega} = 0; \\ \sigma \cdot \mathit{curl}\ V|_{\partial\Omega} = f(y), \quad \displaystyle\int_{\partial\Omega} f\ dy = 0; \\ ((V + \beta y) \times \mathit{curl}\ V)_\tau|_{\partial\Omega} = \nabla_\tau\ g(y), \end{cases} \tag{5}$$

and vanishing at infinity $\quad V = \mathcal{O}(|y|^{-1}), \quad \nabla V = \mathcal{O}(|y|^{-2}) \quad$ *as* $\quad |y| \uparrow \infty.$

Going back to the (x,t) variables, Proposition A asserts that a self-similar solution $v_s(x,t) = V(y)/\hat{r}(t)$ exists on a set (the exterior of closed shrinking balls),

$$(6) \qquad \Sigma = \bigcup_{t\in(0,T)} \widehat{\Omega}(t,n), \quad \widehat{\Omega}(t,n) = \{x \in \mathbb{R}^3 \setminus \{0\} : \hat{r}(t) \le |x| < n\ \hat{r}(t)\},$$

where $\hat{r}(t) = \sqrt{T-t}$, $n \in \mathbb{N}$ sufficiently large. Let B_r be a ball in $\mathbb{R}^3$ centred at the origin with radius r. The set Σ naturally suggests dividing $\mathbb{R}^3 \times (0,T)$ into Ω^{I} (the inner region, with "shrinking" spheres $S_{\hat{r}(t)}$ as its boundary), and Ω^{II} (the outer region):

$$(7) \qquad \Omega^{\mathrm{I}} := \bigcup_{t\in(0,T)} B_{\hat{r}}(t), \quad \Omega^{\mathrm{II}} := \mathbb{R}^3 \times (0,T) \setminus \overline{\Omega^{\mathrm{I}}}, \quad S_{\hat{r}(t)} := \partial\widehat{\Omega}.$$

Since $\Sigma \subseteq \Omega^{\mathrm{II}}$, we need to find other solutions $u(x,t)$ to complete the problem.

3. Construction of outer and inner solutions $u(x,t)$. Given that v_s exists on the set Σ as in (6), substituting (2) into (1) and using (4), we have

$$(8) \qquad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = v_s \times \mathrm{curl}\ u + u \times \mathrm{curl}\ v_s - \nabla(u \cdot v_s), \\ \mathrm{div}\ u = 0 \quad \text{in} \quad \Omega^{\mathrm{I}} \cup \Omega^{\mathrm{II}}, \quad u(x,0) = u_0(x) := v_0 - v_s^0, \quad x \in \mathbb{R}^3 \\ \lim_{|x|\uparrow\infty} u(x,t) = 0 \qquad t \in [0,T]. \end{cases}$$

First, we prove in **Lemma B** existence of u to (8) on Ω^{II}, Ω^{II} as in (7). This is done by considering a natural boundary condition to Euler flows, $\sigma \cdot u|_{\partial\widehat{\Omega}} = 0$, then we are able to treat the outer and inner systems separately.

Having obtained an outer solution u^{II}, to find u^{I} on the inner region we impose

$$u^{\mathrm{I}}|_{\partial\widehat{\Omega}} = u^{\mathrm{II}}(x,t) + v_s|_{\partial\widehat{\Omega}}, \qquad p^{\mathrm{I}}|_{\partial\widehat{\Omega}} = p^{\mathrm{II}}(x,t) + \pi_s|_{\partial\widehat{\Omega}},$$

requiring the continuity of velocities, and that of the pressure, across the boundary. Under this assumption, **Theorem C** states that (1) has at least one weak solution $v = v_s + u$ on $\mathbb{R}^3 \times (0,T)$, which is further shown to become unbounded at a single point in space-time.

Remark I. By Lemma 4B [H] and Theorem C, one deduces that the weak solution $v = v_s + u$ has finite energy: $\int_{\mathbb{R}^3} |v(x,t)|^2\ dx \ \le\ \|v_0\|^2_{L^2(\mathbb{R}^3)}$ $\forall\ 0 < t \le T$. (see also [M] for the localised vortex interaction.)

Remark II. Starting from smooth initial data of finite energy, it is demonstrated that the Euler equations on $\mathbb{R}^3$ can develop a singularity as a sum $v = v_s + u$. Although singular scalings often observed in computations are close to that of v_s, it is clear that a full solution v needs not to have any definite self-similar forms.

References

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