Collapse and concentration of vortex sheets in two-dimensional flow

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Configurations of point vortices that collapse to a point in finite time have been known for a long time [1]. Each vortex traces out a logarithmic spiral, the configuration maintaining the same shape while rotating and shrinking, with configuration area proportional to $t - t_c$ before the vortex positions coincide at time t_c. The normalized time required to collapse depends on the argument of the "collapse parameter" $\omega = -\mathbf{i}\dot{z}_j/z_j$, where z_j is the displacement of the jth vortex from the center of vorticity and $\dot{z}_j$ is the vortex velocity; the ratio is the same for all j. The circulations Γ_j of these configurations must satisfy the relation $(\sum \Gamma_j)^2 = \sum \Gamma_j^2$.

In this paper analogous collapsing configurations of point vortices and vortex sheets are described. The velocity of the points of the sheet are determined by the Birkhoff-Rott equation,

$$2\pi\mathbf{i}\frac{d}{dt}\bar{z}(s,t) = \ p.v.\int_0^1 \frac{\Gamma}{z(s,t)-z(\sigma,t)}\ d\sigma \ + \ \sum_j \frac{\Gamma_j}{z(s,t)-z_j(t)}$$

where $z(s,t)$, $s \in (0,1)$ traces out the vortex sheet at time t and $z_j(t)$, Γ_j are the positions and circulations of the point vorticies. The singularity in the integrand may be removed for a certain class of curves, so that the vortex sheet may be discretized into point vortices [2]. The collapse is characterized by a ratio of velocity to displacement from the center of vorticity with a nonzero imaginary part that is the same for all points on the sheet as well as for the point vortices away from the sheet. The configuration rotates while shrinking in size until collapse occurs. Again, there is a restriction on the allowable vorticities. The most basic collapsing configuration has a single vortex sheet of net circulation Γ and a single point vortex of circulation $-\Gamma/2$ (see figure.) Configurations containing several point vortices and several vortex sheets also exist.

A remarkable feature of the evolution of two-dimensional turbulence at high Reynolds numbers is the spontaneous emergence of coherent vortices and extensive filamentary distributions of vorticity [3]. The vortex sheet collapse configurations described in this paper may be relevant to this evolution, since they suggest a mechanism by which filamentary distributions of vorticity can be concentrated into a small area, leading to roll-up of the filament.

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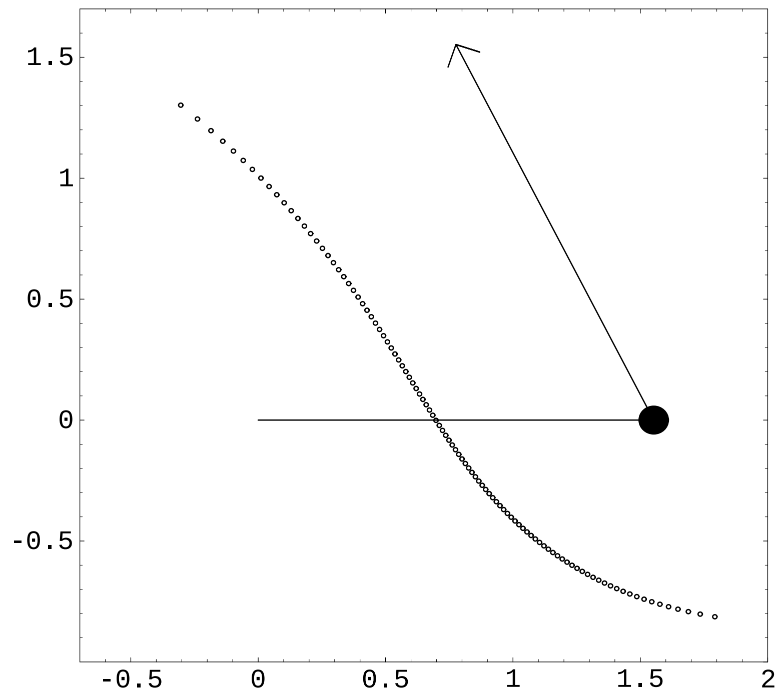

Figure 1: A collapsing configuration of one point vortex with circulation -1 and one positive vortex sheet of total circulation 2; the sheet has been discretized into 100 point vortices of circulation 2/99. The collapse parameter is $\omega = 1 + 0.5\mathbf{i}$. A line segment connects the negative vortex to the center of vorticity, and the angle of the velocity vector indicates the rate at which the configuration shrinks while rotating counter-clockwise.

[1] H. Aref, P. Newton, M. Stremler, T. Tokieda and D. Vainchtein 2002 "Vortex crystals," *Adv. Appl. Mech.* **39** 1–79.

[2] J.-G. Liu and Z. Xin 2000 "Convergence of the point vortex method for 2D vortex sheet," *Math. Comp.* **70** 595 – 606.

[3] N. K.-R. Kevlahan and M. Farge 1997 "Vorticity filaments in two-dimensional turbulence: creation, stability and effect," *J. Fluid Mech.* **346** 49 – 76.