A new calculus for two dimensional vortex dynamics

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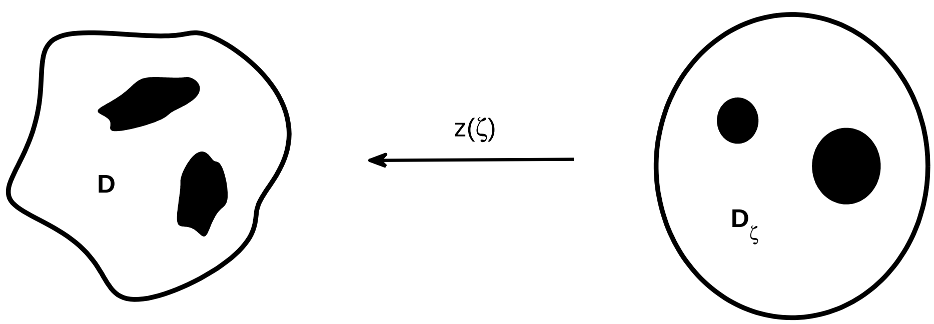
The study of vortex motion in the presence of impenetrable walls is an important one in applications and dates back at least to the work of E.J. Routh in 1881 who studied point vortex motion in simply connected domains. Later, in 1941, C.C. Lin wrote two short papers elucidating what happens if the fluid domain is multiply connected – an ocean basin, for example, with an array of islands. Lin showed that the Hamiltonian structure of the dynamics persists in this case. His mathematical approach, however, was non-constructive.

In the present work, we show how to make Lin's observations constructive and produce a versatile scheme for computing the motion of point vortices in two dimensional domains with complex geometries. Perhaps more significantly, by introducing an important (but little known) special function from classical function theory, we show that it is possible to devise a new "calculus" for ideal flows in general planar domains. The calculus is built around this single special function (it is known as the *Schottky-Klein prime function*). It is a function of two variables, which we denote by $\omega(.,.)$. In terms of it, an array of remarkably compact formulae for solutions to basic problems in inviscid fluid dynamics in the plane (and on the sphere) can be written down in closed form.



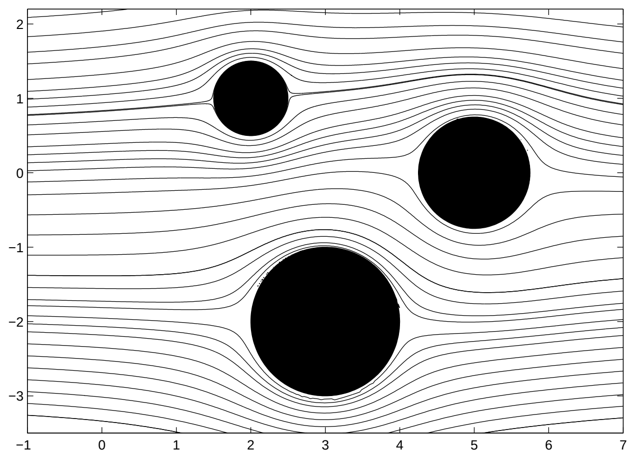
Our approach rests on the remarkable fact, that is an extension of the Riemann mapping theorem, that any multiply connected domain (bounded or unbounded) is the image, under some conformal map $z(\zeta)$, of a bounded multiply connected *circular region* D_ζ. A circular domain is one whose boundaries are all circular. By virtue of this theorem, such domains constitute a *canonical class* of multiply connected domains. The figure above illustrates the triply connected case.

Clearly, the only parameters characterizing the circular domain D_ζ are the centres and radii of its circular boundaries. It is in terms of this minimal set of data that one can define the Schottky-Klein prime function associated with D_ζ. Armed with this single special function, an array of results can be found. These include (a) simple formulae for uniform flow past multiple obstacles; (b) the Kirchhoff-Routh path functions for N-vortex motion in arbitrary domains; (c) the complex potential for the flow generated by an assembly of fluid stirrers; (d) a new formulation of contour dynamics for the evolution of uniform vortex patches in complex geometries and (e) the extension of all these results to the surface of a sphere. To illustrate just (a) and (b), for example:

Example (a): The complex potential for uniform flow, with speed U and angle χ, past *any* collection of obstacles can be written in the simple form

$$W_U(\zeta) = Ua\left[e^{i\chi}\frac{\partial}{\partial\overline{\beta}} - e^{-i\chi}\frac{\partial}{\partial\beta}\right]\log\left(\frac{\omega(\zeta,\beta)}{|\beta|\omega(\zeta,\overline{\beta}^{-1})}\right)$$

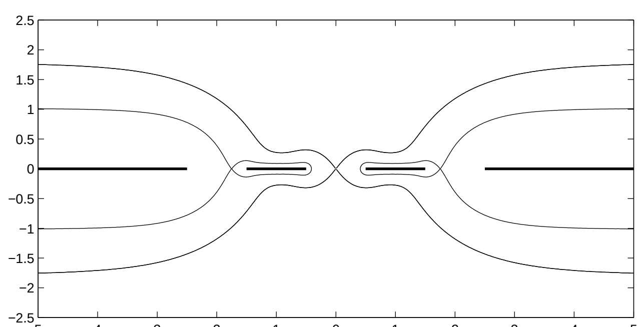
where $\zeta = \beta$ maps to infinity. For example, plotting contours of $\mathrm{Im}[W_U(\zeta)]$ easily produces the instantaneous streamlines for uniform flow past 3 cylinders:



Example (b): The Hamiltonian (or Kirchhoff-Routh path function) governing the complex position $\alpha(t)$ of a single point vortex, with circulation Γ, in a multiply connected domain can be written explicitly as

$$H(\alpha,\overline{\alpha}) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{1}{\alpha^2}\frac{\tilde{\omega}(\alpha,\alpha)\tilde{\omega}(\alpha^{-1},\alpha^{-1})}{\omega(\alpha,\overline{\alpha}^{-1})\omega(\alpha^{-1},\overline{\alpha})}\right|.$$

The critical trajectories (separating regions of qualitatively different dynamical behaviour) of a vortex approaching 3 gaps in an infinite straight wall can, for example, be readily plotted by analyzing contours of this explicit function:



The Hamiltonians for general N-vortex motion can be treated similarly. In this way, Lin's 1941 theory of point vortex motion in multiply connected domains is rendered constructive.

This paper will survey all of these recent developments and give a systematic derivation of this new "calculus" for two dimensional vortex dynamics.