Sound radiation by vortex motions

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Turbulent motion of a flow in certain finite spatial region excites acoustic waves outside this region. In the case of a weakly compressible flow, this sound field is such as if it were generated by the static distribution of acoustic quadrupoles whose instantaneous power per unit volume is given by the relationship $T_{ij}(\boldsymbol{r},t) = \rho_0 v_i^{\mathrm{T}}(\boldsymbol{r},t) v_j^{\mathrm{T}}(\boldsymbol{r},t)$, where ρ_0 is the average density and $v_i^{\mathrm{T}}(\boldsymbol{r},t)$ are the components of the velocity of flow in turbulent region V (see, e.g., [1]). It was shown that, with allowance for incompressibility of flow in volume V and the use of the Kolmogorov–Obukhov hypothesis and a number of simplifying hypotheses for splitting space-time correlators follows that both average energy flux density and acoustic power are proportional to $\sim M^5$, where $M = \dfrac{\sqrt{\langle \boldsymbol{v}^2(\boldsymbol{r},t)\rangle}}{c_0}$ is the *Mach number* (significantly smaller than unity).

We note that this result can be explained purely hydrodynamically, by analyzing vortex interactions in weakly compressible medium [2, 3]. The simplest sound-radiating vortex systems are the pair of vortex lines (radiating cylindrical waves) and the pair of vortex rings (radiating spherical waves).

Sound radiation by vortex lines

Consider two parallel vortex lines separated by distance $2h$ and characterized by equal intensities $\kappa = \dfrac{1}{2}\pi i \sigma$, where i is the vorticity (the size of the vortex uniformly distributed over the area of infinitely small section σ), so that the circulation about each of vortex line is $\Gamma = 2\pi\kappa$. We will call these vortex lines simply vortices. In a noncompressible flow, these vortices revolve with angular velocity $\omega = \dfrac{\kappa}{2h^2}$ around the center of the line connecting these vortices (see, fig. 1a and e.g., [4]).

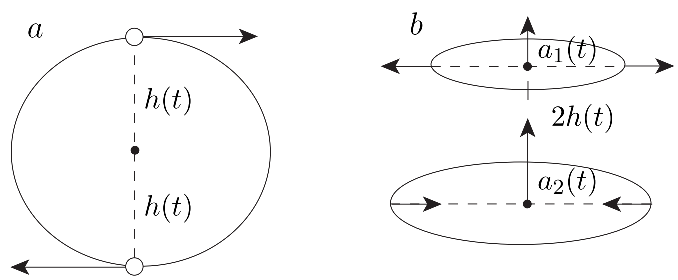

Figure 1. Schematics of dynamics of a) vortex lines and (b) vortex rings

Select the coordinate system with the origin at a fixed point and z-axis along the vortex line. In this coordinate system, velocity potential $\varphi_0(\boldsymbol{r},t)$ assume the form $\varphi_0(\boldsymbol{r},t) = ik\ln\left[r^2 e^{2i\theta} - h^2 e^{2i\omega t}\right]$. Here $r\exp i\theta$ is the radius-vector of the observation point.

According to the Bernoulli equation, local velocity pulsations must produce the corresponding pressure pulsations; in the case of weakly compressible medium, these pressure pulsations will propagate at large distances as sound waves with the sound intensity (energy radiated per unit time) $I = 2\pi^2 \rho_0 M^4 \dfrac{\kappa^3}{h^2}$.

The radiated energy must coincide with the interaction energy of vortices. The interaction energy can vary only at the expense of varying the distance between vortices ($h = h(t)$), because the circulation remains intact due to the fact that we consider nonviscous medium. As result we obtain $h(t) = h_0 \left[1 + 6\pi M_0^4 \omega_0 t\right]^{1/6}$.

Sound radiation by vortex rings

In a noncompressible flow, a vortex ring of intensity κ causes the flow to move with a velocity, which follows from the the *Biot–Savart law* (see, e.g. [4]). Let now we have two vortex rings of equal intensities and equal radii a_0 at distance $2h_0$. In this case, the front ring will increase in size, while the rear ring will decrease and pursue the front ring (fig. 1*b*). At certain instant, it will penetrate through the front ring, and the rings switch places. This phenomenon is called the *game of vortex rings*. In a weakly compressible flow, these motions of rings produce the spherical acoustic waves. If parameter $\gamma = h_0/a_0 \ll 1$, the rings interact actively and

$$a_1(t) = a_0 \left(1 + \gamma \sin(\omega t)\right), \quad a_2(t) = a_0 \left(1 - \gamma \sin(\omega t)\right),$$

$$h(t) = h_0 \cos(\omega t),$$

where $\omega = \dfrac{\kappa}{2h_0^2}$, as in the above case of vortex lines.

The potential in the wave zone has the form

$$\varphi(r, \theta) = \frac{2\kappa}{3} \left(\frac{\kappa}{2h_0 c_0}\right)^2 \frac{a_0}{r} \left(1 - 3\cos^2\theta\right) \exp\left\{2i\left(\omega t - \frac{\omega r}{c_0} - \frac{3\pi}{4}\right)\right\},$$

so that vortex rings radiate energy as a quadrupole and the total energy radiated per unit time is given by the expression $I = \dfrac{64\pi}{45} \dfrac{\kappa^3 a_0^2}{h_0^3} \left(\dfrac{\kappa}{2h_0 c_0}\right)^5 \rho_0$.

Thus, intensity of sound radiated by a pair of vortex rings is proportional to M^5, which agrees with estimates obtained in [1].

[1] Lighthill, J., Proc. Roy. Soc. **A.211**(1107), 564–587, 1952; **A.222**(1148), 1–32, 1954; **A.267**(1329), 147–182, 1962.

[2] Klyatskin, V.I., Izvestiya AN SSSR, Mekhanika Zhidkosti i Gaza No. 6, 87–92, 1966 (in Russian).

[3] Klyatskin, V.I., *Stochastic Equations Through the Eye of the Physicist*, Elsevier, Amsterdam, 2005.

[4] Milne-Thomson, L.M., *Theoretical Hydrodynamics* 5-th edition, Dover Publication, Inc., New York, 1996.