Dynamics of vortex rings in viscous fluids*

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The work of many authors on vortex rings has been a great inspiration to us. A common approach is to take expressions like the velocity of a vortex ring in an ideal fluid

$$V = (\Gamma / 4\pi R)[\ln(8R/a) - \beta] \qquad (1)$$

and take the logarithmic term as a constant, and replace the ring radius R by the radius of the gun R_0. This coupled with the "slug model" ($\Gamma_s = L^2 / 2T$) for the circulation allows them to deduce all sorts of expressions for various desired quantities by dimensional analysis.

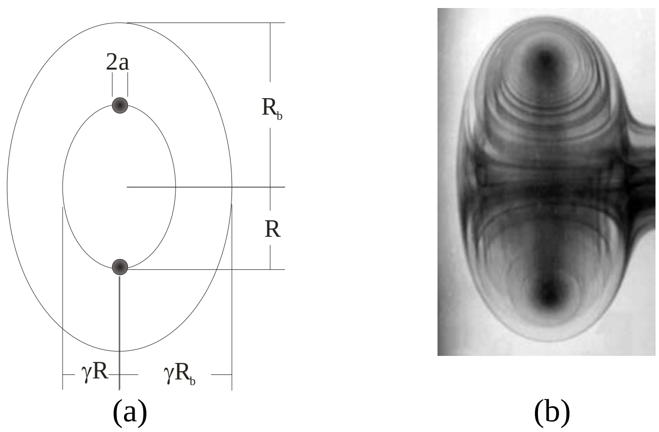

Figure 1 (a) Sketch of a thin vortex ring of core radius a, ring radius R and bubble radius R_b. γ is the ratio of semi minor to semi major axes. The direction of motion is horizontal. (b) Photograph of a ring that has just passed through a sheet of tracer, which causes the apparent jet on the right. The comparison with (a) is clear.

However it does not take much experimentation to note that vortex rings can have radii both bigger and smaller than the gun radius R_0. Neglect of the logarithmic term means that information on the core size a and the vortex core structure parameter β has been lost. Eq. (1) is the result for a perfect fluid, and it has been experimentally verified by Rayfield and Reif (1964) in superfluid helium-4 at 0.28 K.

The key question we have worked on is what modifications to (1) are needed to describe the velocity of a vortex ring in a viscous medium such as water? Indeed such an expression has been introduced by Saffman (1970) for a Gaussian vorticity distribution in the core:

$$V = \frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{a} - \beta\right) \qquad (2)$$

and where he finds

$$a \to \sqrt{4\nu T},\ \beta = 0.558 \qquad (3)$$

where T is the stroke time. Our paper (Sullivan *et al.*2008) establishes the fact that (2) and (3) agree with experiment and that, most astonishing, once the ring emerges from the gun, it does not grow further. We have found this result totally counter-intuitive, and have worked hard to disprove it.

The ring radius R is given by equating the volume of the inner ellipsoid in figure 1 to the volume of water displaced by the piston and leads to the expression

$$R = \left(3R_0{}^2 L / 4\gamma\right)^{1/3} \tag{4}$$

which agrees very well with our measurements. γ is independent of velocity as given by inviscid theory, and we obtain it simply by photography. Similar arguments give an expression for the bubble size (see figure 3):

$$R / R_b = (\gamma(1+k)\Lambda / 3\pi)^{1/3} \tag{5}$$

We then use two different expressions for conservation of momentum, which seem reasonable to use in a viscous fluid, since momentum is conserved in dissipative systems in classical mechanics. This allows us to derive expressions for the circulation in terms of piston velocity $V_p = L/T$.

$$\Gamma = R_0{}^2 L V_p / R^2 = R_0{}^2 L^2 / R^2 T \tag{6}$$

which we believe is more accurate than the slug model, but gives comparable results in many cases. Note this means the circulation can be given in terms of the stroke time T, stroke length L and ring radius R (which itself comes from L in 4). This means that an experimenter can set the circulation in terms of gun characteristics only.

With (4) in hand it is not hard to derive an expression for the velocity of the ring, which turns out to be proportional to the piston velocity.

$$V = \frac{\Gamma}{4\pi R}\left(\ln\frac{8R}{a} - 0.558\right) = \frac{\gamma V_p}{3\pi}\Lambda \tag{7}$$

and Λ is the log factor in brackets. Eq. (7) predicts

$$V / V_p = \gamma\Lambda / (3\pi) \simeq 0.30 \tag{8}$$

which agreed well with our results using an earlier gun design. Imagine our surprise when we discovered our new gun gives $V / V_p \simeq 0.63$. The origin of this problem appears to be Kelvin waves on the cores of the rings at formation.

The formulae we have developed allow us to know exactly what the properties of the projectile vortices are. If we set the stroke length L and stroke time T on each gun (they do not have to be identical) then the formulae given above allow us to know the core parameter (3), the ring radius (4), the bubble size (5) and the circulation (6). The velocity is given by (7) . After the scattering event, we can obtain the circulation from (1) by measuring the velocity and radius of the product rings (and core size if needed) photographically.

These considerations have allowed us to construct a vortex ring scattering chamber. We have two identical vortex guns mounted on the circumference of a calibrated circle. We can set the stroke time T and stroke length L to produce rings of desired properties. Data is obtained photographically. To the best of our knowledge this project offers a most sophisticated experimental apparatus for research on vortex dynamics. The characterization of rings by the formulae given above allows rapid exploration of a wide variety of initial collision conditions photographically. The initial conditions can be set and reproduced to machine shop accuracy. We are looking forward to productive use of this apparatus, and the kind of insight one can gather from vortex ring collision experiments.

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