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Self-similar collapse of 2D and 3D vortex filament models

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Search for finite-time singularity (FTS) for Euler's equation (or the inviscid limit of the Navier-Stokes equations) is one of the most challenging topics in mathematical fluid dynamics. It is not only of academic interest, but the location, form and dynamics of singularities could provide crucial models and information of complex fluid flows such as turbulence and vortex reconnection.

As a specific candidate initial condition for a finite-time singularity in Euler's equation, the vortex dodecapole[1]−[3], the superposition of three equal-strength, orthogonal, vortex quadrupoles, has been proposed. The flow under this initial condition is highly symmetric, and it tends to develop to collapse in a self-similar fashion at the origin of the coordinates. The numerics of pseudospectral method shows that vorticity scales $(t_c - t)^{-1}$ for the flow. Later, Pelz verified the scaling by investigating the motion of the vortex quadrupoles with the vortex method[4].

In this paper, we shall examine perhaps the simplest model of the vortex dodecapole in which we replace the vortex tubes with *straight* vortex filaments of infinitesimal thickness, and the entire motion is monitored by tracking the motion of a representative point on one vortex filament. Let us call this model the filament dodecapole. We attempt to obtain the similarity solution of the model equation using the same procedure for the similarity solution of 2D point vortex systems[6].

The filament dodecapole, shown in figure 1, has three orthogonal vortex quadrupoles parallel to the x, y and z axes, which we name x-, y- and z-quadrupole, respectively. We locate a representative point at the intersection of one of the z-quadrupole with the plane of $z = 0$ in the first quadrant of the xy-plane, and call it $\mathbf{P} = (x_0, y_0, 0)$, $(x_0 > 0, y_0 > 0)$. First we write down the velocity at (x, y) in the xy-plane($z = 0$) induce by z-quadrupole with filaments parallel to the z axis and go through $(x_0, y_0), (-x_0, y_0), (x_0, -y_0), (-x_0, -y_0)$. Letting the strength $\Gamma = \pm 2\pi$, the horizontal and vertical velocity components of the induced velocity at $\mathbf{P}(x_0, y_0, 0)$are obtained by making use of the high symmetry of the system.

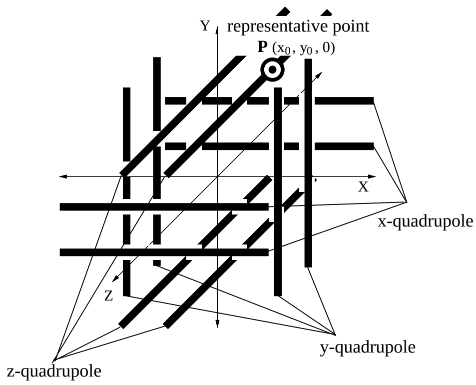

Figure 1: The filament dodecapole

The final result of the equations are

$$u = \dot{x}_0 = \underbrace{+\frac{1}{2y_0} - \frac{y_0}{2(x_0^2 + y_0^2)}}_{\text{from z-quadrupole}} \underbrace{-\frac{2x_0}{(y_0 - x_0)^2 + x_0^2} + \frac{2x_0}{(y_0 + x_0)^2 + x_0^2}}_{\text{from y-quadrupole}} \tag{1}$$

$$v = \dot{y}_0 = \underbrace{-\frac{1}{2x_0} + \frac{x_0}{2(x_0^2 + y_0^2)}}_{\text{from z-quadrupole}} \underbrace{+\frac{2y_0}{(y_0 - x_0)^2 + y_0^2} - \frac{2y_0}{(y_0 + x_0)^2 + y_0^2}}_{\text{from x-quadrupole}}. \tag{2}$$

the above equations provide an autonomous dynamical system for the motion of $x_0(t)$ and $y_0(t)$, and hereafter we shall concentrate on these equations.

We seek the similarity solutions for (1) and (2) by the following procedure. First we assume that all the variables develop with the same time dependence, $f(t)$, and substitute

$$x_0(t) = f(t)\xi, \qquad y_0(t) = f(t)\eta. \tag{3}$$

into the equations (1) and (2). By separating variables into the time and space parts, we obtain two equations,

$$f\dot{f} = c \tag{4}$$

$$\begin{aligned} & \frac{1}{\xi}\left[\frac{1}{2\eta} - \frac{\eta}{2(\xi^2+\eta^2)} - \frac{2\xi}{(\eta-\xi)^2+\xi^2} + \frac{2\xi}{(\eta+\xi)^2+\xi^2}\right] \\ = \; & \frac{1}{\eta}\left[-\frac{1}{2\xi} + \frac{\xi}{2(\xi^2+\eta^2)} + \frac{2\eta}{(\eta-\xi)^2+\eta^2} - \frac{2\eta}{(\eta+\xi)^2+\eta^2}\right]. \end{aligned} \tag{5}$$

where c is a real constant to be determined consistently with other variables.

After some labor, we can obtain the similarity solutions of (4) and (5). We will discuss the stability of the similarity solutions and the physical meaning of them. Some realistic features of the solution can been seen when we look at the motion of particles around the solution.

References

[1] O. N. Boratav and R. B. Pelz, "Direct Numerical Simulation of Transition to Turbulence from a High-Symmetry Initial Condition", Phys. Fluids **6** (1994) 2757.

[2] O. N. Boratav and R. B. Pelz, "On the Local Topology Evolution of a High-Symmetry Flow", Phys. Fluids, (1995).

[3] O. N. Boratav and R. B. Pelz, "Locally isotropic pressure Hessian in a high-symmetry flow," Physics of Fluids, **7** (1995) 895.

[4] R. B. Pelz, "Locally self-similar, finite-time collapse in a high-symmetry vortgex filament model," Phys. Rev. E, **55** (1997) 1617–1626.

[5] R. B. Pelz and Y. Kimura, "Self-similar, finite-time collapse of the straight vortex-filament dodecapole" in RIMS Kokyuroku **1326**, *Rich Pelz' Contributions to Fluid Dynamics* (2003) 10–13, http://hdl.handle.net/2433/43207

[6] Y. Kimura. "Similarity solutions of two-dimensional point vortices" *J. Phys. Soc. Japan*, **56** (1987) 2024–2030.

[7] Y. Kimura. "Parametric motion of complex-time singularity toward real collapse" *Physica*, **D46** (1990) 439–448.