New integrable problem of motion of point vortices on a sphere

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Consider $2N$ point vortices with circulations $\Gamma_1, \ldots, \Gamma_N, -\Gamma_1, \ldots, -\Gamma_N$. The equations of motion for this system of vortices admit the following invariant manifold

$$\theta_i + \theta_{N+i} = \pi, \varphi_{N+i} = \varphi_i + \pi 2\pi. \quad (1)$$

Thus, if at the initial time the vortices with opposite circulations are diametrically opposite, they will remain so forever. Such a pair of vortices can be thus considered as a single vortex structure that we call *antipodal vortex*. The equations of motion for antipodal vortices can be written as

$$\dot{\theta}_k = -\frac{1}{2\pi R^2}\sum_{i\neq k}\Gamma_i\frac{\sin\theta_i\sin(\varphi_k-\varphi_i)}{\sin^2\gamma_{ik}},$$

$$\sin\theta_k\dot{\varphi}_k = -\frac{1}{2\pi R^2}\sum_{i\neq k}\Gamma_i\frac{\cos\theta_k\sin\theta_i\cos(\varphi_k-\varphi_i)-\sin\theta_k\cos\theta_i}{\sin^2\gamma_{ik}}. \quad (2)$$

Here R is the radius of the sphere, γ_{ik} is the angle between the radius vectors from the spheres center to the positions of the vortices with numbers i and k, and θ_i, φ_i are the spherical coordinates of the $i-$th vortex. These equations are Hamiltonian and admit integrals of motion generically inherent to point-vortex systems. We have obtained some results on reduction and integrability of equations Sp8. The motion of two and three antipodal vortices is explored in greater detail, new periodic motions are found and a complete bifurcation analysis of the problems is performed.

References

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