Relative equilibria of point vortices

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The problem of finding relative equilibria of interacting point vortices is both of intrinsic mathematical interest and of interest in applications to superfluids such as He II and BECs. Through "designularization" it also suggests solutions with smooth vorticity that may exist for the 2D Euler equations. For point vortices the problem of relative equilibria amounts to a problem in algebraic geometry. One is led to study the polynomials that have the vortex positions as their roots alongside the equilibrium configurations. The paper will survey known results for identical point vortices, obtained by analysis and numerical computation. The quite considerable development in our understanding of the solution space of these simple problems relative to the 1978 "catalog" of Campbell & Ziff will be traced.

1 Physical motivation

Relative equilibria of point vortices are closely realized in superfluids and plasmas. In ordinary fluids they provide approximations to observed structures such as the vortex tripole, vortex streets, composite cores of hurricanes and much else. See Fig.1. Other systems of particles interacting through a fluid also form patterns corresponding to relative equilibria, and these have been thought of as analogues of the vortex patterns since the time of Kelvin. A review of what was known about this problem until recently appeared in [4].

2 Mathematical interest

For identical vortices the problem of relative equilibria reduces to a problem of algebraic geometry, *viz* the solution of the following system of N equations in N complex unknowns z_α, $\alpha = 1, \ldots, N$:

$$z_\alpha^* = \sum_{\beta=1}^{N}{}' \frac{1}{z_\alpha - z_\beta}. \tag{1}$$

Here the asterisk means complex conjugation and the prime on the summation sign reminds us to skip over the singular term $\beta = \alpha$. The angular frequency of rotation of the configuration, ω, and the common value of the circulation, Γ, have been scaled such that $2\pi\omega/\Gamma = 1$.

Classical analytical solutions to Eqs.(1) include regular polygons, both open and centered, going back to Kelvin and Thomson [15], vortices on a line [14], two nested, regular polygons ("double rings") [10], again both open and centered, and "triple rings" [3], also both open and centered. If one nests regular n-gons, one can nest an arbitrary number, whether in an open or centered configuration. However, once one departs from configurations of high symmetry, analytical understanding of relative equilibria decreases dramatically. In particular, the completely asymmetric configurations, first found in [5], still defy analytical understanding.

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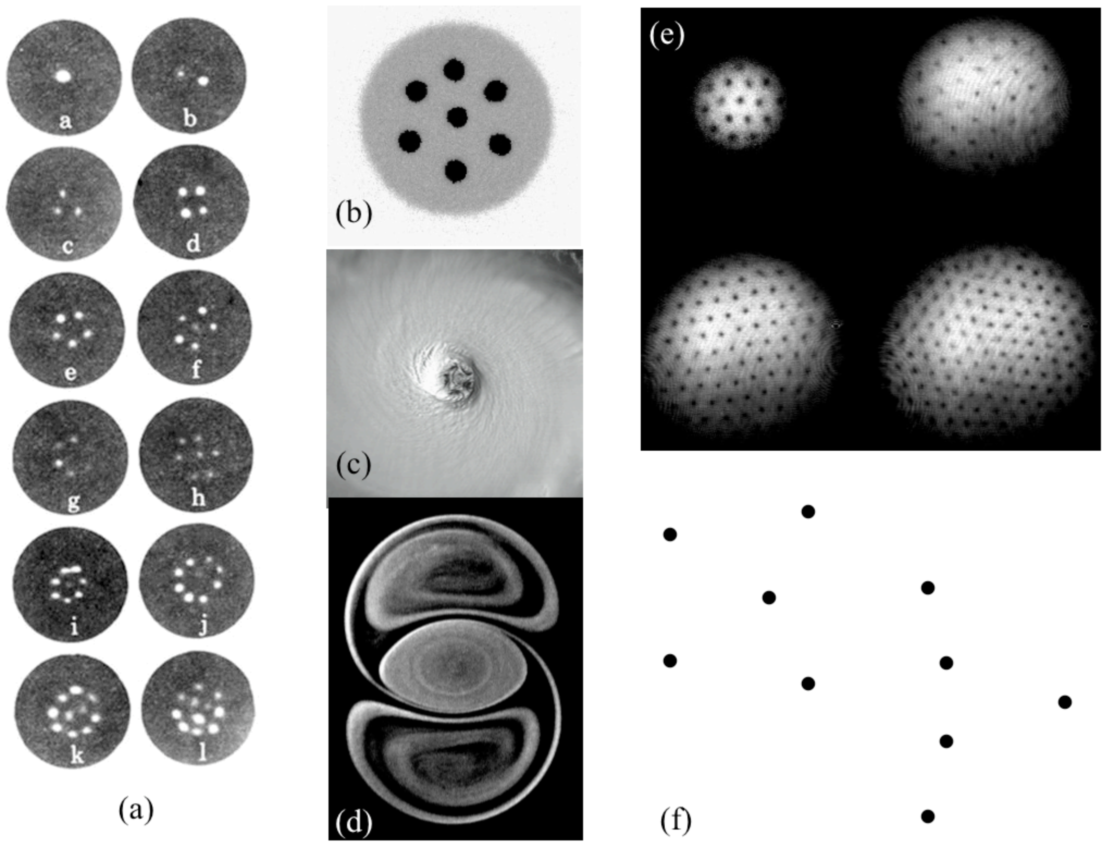

Figure 1: Various relative equilibrium patterns: (a) vortices in He II [16]; (b) electron density peaks in plasma [9]; (c) hurricane core with centered pentagon of meso-vortices [13]; (d) vortex tripole [12]; (e) vortex pattern in BEC [11]; (f) two little known configurations of five identical vortices.

One useful device in studying these relative equilibria is the *generating polynomial* by which we mean the polynomial $P(z)$ of degree N that has the z_α from Eqs.(1) as its roots, *viz*

$$P(z) = (z - z_1)(z - z_2)\dots(z - z_N). \tag{2}$$

The interplay between relative equilibria and properties of the generating polynomials appears to be a very interesting area of mathematical physics [1, 2]. In some cases the resulting polynomials are well-known classical polynomial families and this identification provides a complete solution to the problem [6, 14].

The 1978 catalog of Campbell & Ziff [7] (henceforth referred to as CZ), unfortunately unpublished (but see [8]) has been re-examined by numerical computations. A compact algorithm for ascertaining linear instability is available. CZ focused mainly on states that are not linearly unstable. Thus, for $N = 7$ CZ give just two relative equilibria, the open regular heptagon and the centered regular hexagon. Our compilation includes additional unstable configurations for a total of 11 configurations found so far. Similarly, for $N = 8$ CZ lists only the centered, regular heptagon whereas we now know 19 relative equilibria, some of them analytically. The case $N = 8$ includes the smallest known asymmetric relative equilibrium [5], which one can count as two since both it and its mirror image are equilibria although they cannot be produced from one another by the known point symmetries. For $N = 9$ CZ give four equilibria, two of which are unstable. We have found at least 23. Even for small N, e.g., $N = 5, 6$, we find surprises. A survey of the new catalog will be given.

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