A moment model for the motion of a dipole in two-dimensional incompressible flows

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A simple model for the motion of a dipolar vortex in two-dimensional inviscid flows is proposed. The motions of a dipole is expressed by a set of ordinary differential equations for the dipole moment, length scale, and centroid which are derived from the conservation of mass and momentum. Using the model, we analyze behaviors of a dipole in various background flows. Comparing results of the model with numerical simulations of the vortex method, remarkable agreements are obtained, and the validity of the model is confirmed.

I. INTRODUCTION

A dipolar vortex is one of the basic vortical structures of two-dimensional flows. The dipole consists of a pair of equal and oppositely signed vortices. In the past, dipoles have been observed in nature such as the atmosphere or the ocean.[1,2] The dipole translates by a self-propelling velocity. This motion is considered to transport fluid mass, heat, and so on. Hence it is important to investigate evolution of a dipole in a background flow. Motions of a dipole in a strain flow were studied by numerically and experimentally.[3,4] However, studies of a dipole in other background flows seem to be few in literatures.

In this work, a simple model for a dipole is proposed in order to investigate a dipole motion in various background flows. The dipole is characterized by three variables, the centroid, dipole moment and length scale. The motions of the dipole is governed by a set of ordinary differential equations for these variables. We name this model as a moment model because the dipole moment represents main features of the dipole, the strength and translating direction.

II. FORMULATION OF THE MOMENT MODEL

We consider the motion of a dipole in the two-dimensional incompressible inviscid flow in an unbounded domain. The dipole consists of a pair of equal and opposite counter-rotating vortices with circulations $\Gamma_\pm$ in finite regions $\Omega_\pm$, where subsctips show the sign of the vorticity. Note that $\Gamma_+ = \Gamma$ and $\Gamma_- = -\Gamma$ so that the total circulation of the dipole equals zero. The schematic drawing of the dipole is shown in Fig. 1. The centroid and the area of each vortex are given by $\boldsymbol{x}_\pm = \int_{\Omega_\pm} \boldsymbol{x}\mathrm{d}S$ and $S_\pm = \int_{\Omega_\pm} \mathrm{d}S$, respectively. We consider only the case $S_+ = S_-$ in this study.

In the moment model, the dipole is characterized by three time-dependent variables as defined below: the centroid $\boldsymbol{x}_\mathrm{d}$, dipole moment $\boldsymbol{\mu}$, and length scale a.

$$\boldsymbol{x}_\mathrm{d} = (\boldsymbol{x}_+ + \boldsymbol{x}_-)/2, \tag{1}$$

$$\boldsymbol{\mu} = \int_\Omega \boldsymbol{x}\omega(\boldsymbol{x})\mathrm{d}S \times \boldsymbol{e}_z = \Gamma(\boldsymbol{x}_+ - \boldsymbol{x}_-) \times \boldsymbol{e}_z, \tag{2}$$

$$a = |\boldsymbol{x}_+ - \boldsymbol{x}_-|, \tag{3}$$

where $\Omega = \Omega_+ + \Omega_-$ and $\boldsymbol{e}_z$ is the unit vector in the z-direction. Equations (2) and (3) give the magnitude of the dipole moment, $|\boldsymbol{\mu}| = \Gamma a$.

From conservation of mass and momentum and Kelvin's circulation theorem, a set of ordinary differential equations for the dipole moment with length scale is derived as

$$\frac{d\mu_x}{dt} = -\mu_x \left.\frac{\partial U_x}{\partial x}\right|_{\boldsymbol{x}_\mathrm{d}} - \mu_y \left.\frac{\partial U_y}{\partial x}\right|_{\boldsymbol{x}_\mathrm{d}}, \tag{4}$$

$$\frac{d\mu_y}{dt} = -\mu_x \left.\frac{\partial U_x}{\partial y}\right|_{\boldsymbol{x}_\mathrm{d}} - \mu_y \left.\frac{\partial U_y}{\partial y}\right|_{\boldsymbol{x}_\mathrm{d}}, \tag{5}$$

$$a = \mu(t)/\Gamma, \tag{6}$$

where $\boldsymbol{U} = (U_x, U_y)$ is the velocity of a background flow. The evolution equation for the centroid is also expressed as

$$\frac{\mathrm{d}\boldsymbol{x}_\mathrm{d}}{\mathrm{d}t} = \boldsymbol{U}(\boldsymbol{x}) + \boldsymbol{u}_\mathrm{self}(t), \tag{7}$$

where $\boldsymbol{u}_\mathrm{self}$ is the self-propelling velocity of the dipole parallel to $\boldsymbol{\mu}$. Calculations of the family of vortex pair by Pierrehumbert[5] showed that $|\boldsymbol{u}_\mathrm{self}|$ varies with a. Therefore we set $|\boldsymbol{u}_\mathrm{self}|$ to vary according to changes in a smoothly.

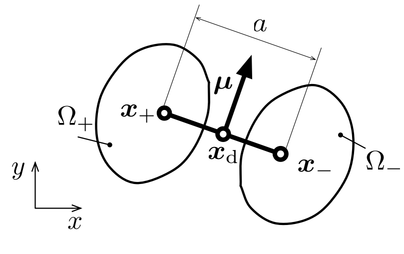

FIG. 1: Schematic drawing of a dipole. Shown are two vortices with regions $\Omega_\pm$. The centroid of each vortex is denoted by $\boldsymbol{x}_\pm$.

III. VALIDATION OF THE MOMENT MODEL BY NUMERICAL SIMULATIONS

As a test calculation we consider evolution of a dipole in a strain flow. In calculations of the model, the system of equations (5), (6), and (7) are solved numerically. In order to demonstrate the validity of the moment model, the results are compared with numerical simulations of vortex methods.[6] As an initial vorticity distribution for the dipole we use the Lamb-Chaplygn dipole[7,8] both in the moment model and the vortex method.

We consider two initial configurations that the dipole is located in opposite or same direction with the strain flow as shown in Fig. 2. calculatio results of the moment model and the vortex method are drawn together in Fig. 3 and 4. Thick lines show the dipole moment (arrow) and the length scale like in Fig. 1. The point at the intersection of the dipole moment with the line of the length scale indicates the position of the centroid of the model. Thin lines are contours of vorticity calculated by the vortex method. From vorticity contour in Fig. 3, we observe that vortices of the dipole are separated by the strain flow. On the other hand, figure. 4 shows that the dipole is elongated and becomes deformed to the head-tail structure.[3] In both cases, the centroid of the model are located on the vorticity center. Additionally changes in the length scale correspond to separation or elongated behavior. In calculations with other background flows, for example in a rotational or a shear flow, similar behaviors are observed. Therefore, we confirm the moment model is valid to represent the motion of a dipole in a background flow.

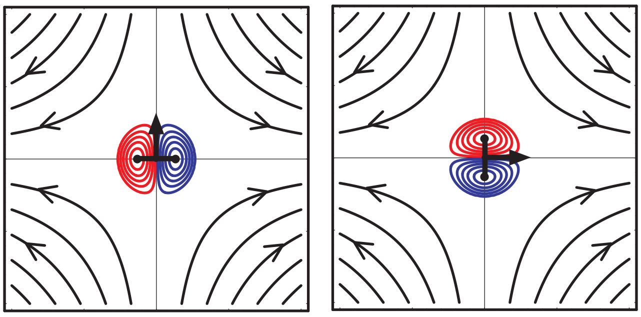

FIG. 2: Initial configurations of a dipole in a strain flow. Streamlines of a strain flow are drawn.

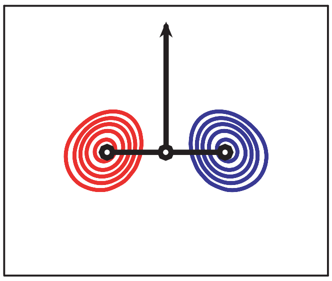

FIG. 3: Evolution of the dipole in a strain flow in the case of that the dipole located in the opposite direction with the strain.

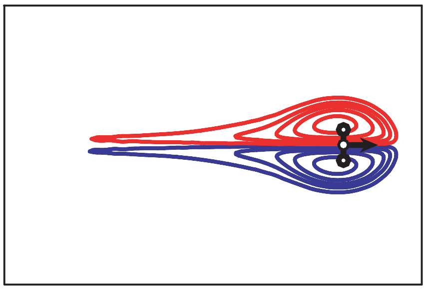

FIG. 4: Evolution of the dipole in a strain flow in the case of that the dipole located in the same direction with the strain.

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