Vortex Dynamics of Wakes: Analysis of the "Domm system"

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Vortex wakes are very common. The structure and dynamics of a wake can have a significant effect on the object forming it as well as on other objects with which the wake interacts. In order to gain a better understanding of the two-dimensional nonlinear dynamics in the vicinity of a vortex street wake, we analyze what we call the *Domm system*, four vortices in a periodic strip, all of the same absolute magnitude, two of either sign.

1. Introduction

In 1911 the seminal work of von Kármán provided the first analytic theory of the structure and stability of a vortex street wake [8]. He first showed that a double row of vortices, with opposite vortices in the two rows, will only propagate downstream if it is either symmetric or staggered. He then considered perturbations of these configurations and, after a first wrong attempt, arrived at his famous criterion for the absence of linear instability of a vortex street:

$$\sinh(\frac{\pi b}{h}) = 1. \tag{1}$$

Here b is the separation of the two vortex rows and h is the intra-vortex spacing within each row. Kármán's criterion leads to $b/h = \frac{1}{\pi}log(1+\sqrt{2}) = 0.28055\ldots$. Later Dolaptschiew [3, 4] and Maue [9] generalized his result to encompass oblique vortex streets. Several workers – we mention Dolaptschiew and Kochin, in particular – established that even if the vortex street was not unstable to linear order, it was unstable when further nonlinear interactions were included. In 1956 Domm considered the dynamical system of four point vortices, all of the same absolute magnitude but two of either sign, in a periodic strip [5]. While the problem of two opposite vortices in a periodic strip suffices to determine when a vortex street will propagate downstream and what the velocity of propagation will be, four vortices are required to address stability. Domm carried the expansion about the vortex street relative equilibrium to second order in the deviations from the street values, and showed that even if (1) is satisfied, the vortex street is still unstable when second order perturbations are included. This elegant analysis, which we wish to deepen and extend, is the reason we call the system of four vortices, all of the same absolute strength, two of either sign, in a periodic strip the *Domm system*.

In general the equations of motion for N point vortices, with strengths Γ_α, $\alpha = 1, 2, \ldots, N$, in a periodic strip of width L are [7]

$$\overline{\frac{dz_\alpha}{dt}} = \frac{1}{2L\mathrm{i}} \sum_{\beta=1}^{N}{}' \, \Gamma_\beta \cot\left[\frac{\pi(z_\alpha - z_\beta)}{L}\right]. \tag{2}$$

The overbar represents complex conjugation. Each z_α represents an entire row of equally spaced vortices with spatial period L, i.e., the vortices of the row are at $z_\alpha + nL$, $n = \pm1, \pm2, \ldots$ and they all have circulation Γ_α. These equations of motion have three general integrals: The components X and Y of the linear impulse Z,

$$Z = X + iY = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha, \tag{3}$$

and the Hamiltonian of the system

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta \log\left|\sin\left(\frac{\pi}{L}(z_\alpha - z_\beta)\right)\right|. \tag{4}$$

2.The Domm system

The Domm system is the special case $N = 4$, $\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = \Gamma$ of the above. Let z_1 and z_2 be the positions of the positive vortices, ζ_1 and ζ_2 the positions of the negative vortices. Analyzing this system on the infinite plane, Eckhardt & Aref [6] showed that the transformation from z_1, z_2, ζ_1, ζ_2 to Z, Z_0, Z_+, Z_- given by

$$\begin{aligned} Z &= \frac{1}{2}(z_1 + z_2 - \zeta_1 - \zeta_2), \qquad & Z_0 &= \frac{1}{2}(z_1 + z_2 + \zeta_1 + \zeta_2), \\ Z_+ &= \frac{1}{2}(z_1 - z_2 + \zeta_1 - \zeta_2), \qquad & Z_- &= \frac{1}{2}(z_1 - z_2 - \zeta_1 + \zeta_2), \end{aligned} \tag{5}$$

is canonical. Since Z is the linear impulse, and thus a constant of the motion, Z_0 is cyclical. Thus, the transformed Hamiltonian depends only on Z_+ and Z_- and contains Z as a parameter. Domm used the variables Z_+ and Z_- in his analysis but did not realize that they represented a canonical transformation valid and useful for all values of these variables, not just for small perturbations around the steady state represented by a vortex street. In terms of the transformed variables the Hamiltonian takes the form

$$H = -\frac{1}{2\pi} \log \left| \frac{\sin\left[\frac{\pi}{L}(Z_+ + Z_-)\right] \sin\left[\frac{\pi}{L}(Z_+ - Z_-)\right]}{\sin\left[\frac{\pi}{L}(Z + Z_-)\right] \sin\left[\frac{\pi}{L}(Z_+ + Z)\right] \sin\left[\frac{\pi}{L}(Z_+ - Z)\right] \sin\left[\frac{\pi}{L}(Z - Z_-)\right]} \right|. \tag{6}$$

We may, thus, study the system both in terms of the four original complex variables z_1, z_2, ζ_1, ζ_2, i.e., in an 8-dimensional space, and in terms of the two variables Z_+ and Z_- with Z as a parameter. The latter representation "lives" in a four-dimensional space, and has H as a conserved quantity. Hence, system trajectories may be represented in a three-dimensional space.

We may also use this representation to find initial conditions with the same values of the integrals Z and H and to explore the vicinity of the Kármán vortex street configuration using such initial conditions. For a vortex street configuration we get with our earlier notation that $Z_+ = -h$, $Z_- = 0$, whereas $Z = -\frac{h}{2} - ib$. Perturbing Z_+ and Z_- arbitrarily and then reconstructing z_1, z_2, ζ_1, ζ_2 from the inverse relations to (5) leaves the value of Z unaltered. To preserve H we need to assure that

$$H(Z_+, Z_-; Z) = H(-h, 0; -\frac{h}{2} - ib), \tag{7}$$

and this equation can be solved for Z_+ and Z_- and the vortex coordinates reconstructed. We may then check that the perturbation in real space of the vortex positions away from the vortex street configuration is not too large. In this way we obtain initial conditions with the same linear impulse and kinetic energy as the vortex street, yet perturbed away from that configuration. We study the numerical evolution of these perturbations by high precision numerical simulations.

When the criterion (1) is satisfied, we find that the vortices initially oscillate many times about the vortex street configuration. These oscillations increase in amplitude until the vortices come so close to opposite-signed partners that they pair up and fly off to infinity. The oscillations are in accord with investigations by Dolaptschiew 70 years ago [3, 4], who derived an approximate dynamical system for perturbations close to the von Kármán vortex street and studied its solutions by numerical hand computations. The phenomenology of vortices from opposite sides of the street pairing up and flying off to infinity is consistent with earlier observations in numerical simulations [1] and experiments [2]. A new observation, reminiscent of the chaotic scattering discussed in [6], is that the pairing up is extremely sensitive to initial conditions, i.e., to the perturbation used. We have codified this by considering which vortices pair, and we find that the pairing can take place with different partners from the opposite row both upstream and downstream. Thus, although the short-time dynamics is basically insensitive to small changes in the initial conditions , the long-time dynamics of pairing is extremely sensitive. This may indicate that the Kármán vortex street configuration is a *chaotic saddle point* of some kind in the phase space of the Domm system. We are currently exploring this question.

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