Dynamics of Vortex Line in Presence of Stationary Vortex

V.E. Zakharov

We study dynamics of infinitely thin vortex line of zero vorticity and unit local induction parameter in a velocity field created by infinitely thin stationary vortex line of vorticity q. If the shape of line is a one-valued function of vertical coordinate z, position of the moving vortex line is described by a complex function

$$\Psi = \Psi(z,t) \quad \Psi = x + iy$$

Function $\Psi(z,t)$ satisfies equation

$$\frac{\partial \Psi}{\partial t} = i \left(\frac{\partial}{\partial z} \frac{1}{\sqrt{1+|\Psi_z|^2}} \Psi_z + \frac{q}{\Psi} \right) \tag{1}$$

If $q > 0$, both vortices rotate in the same direction (co-rotating case); if $q < 0$, vortices rotate in the opposite direction (anti-rotating case). If $q = 0$, the moving vortex is free and Equation (1) is an integrable system equivalent to the focusing Nonlinear Schrodinger equation (NLSE).

Equation (1) is Hamiltonian

$$\frac{\partial \Psi}{\partial t} = i \frac{\delta H}{\partial \bar{\Psi}} \qquad H = \int \left\{ 2(R-1) - q \ln \frac{|\Psi|^2}{|\Psi_0|^2} \right\} dz \tag{2}$$

Here $R = \sqrt{1+|\Psi_x|^2}$. We assume that $|\Psi| \to |\Psi_0|^2$ at $z \to \infty$. Besides H, it has the following constants of motion

$$N = \int \left(|\Psi|^2 - |\Psi_0|^2 \right) dz$$

$$P = i \int |\Psi \bar{\Psi}' - \bar{\Psi} \Psi'| dz$$

Equation (1) has helix-type exact solutions

$$\begin{aligned} \Psi &= A e^{ikz - i\omega t} \\ \Omega &= \frac{k^2}{\sqrt{1+A^2k^2}} - \frac{q}{A^2} \end{aligned} \tag{3}$$

Study of spiral stability with respect to small perturbation

$$\Psi = Ae^{ikz-i\omega t}\left(1+\delta Ae^{-i\omega t+ipz}\right)$$

leads to the following dispersion relation

$$\omega^2 = \frac{p^2}{1+k^2A^2}\left\{\frac{p^2}{1+k^2A^2} - \frac{k^4A^2}{1+k^2A^2} + \frac{2q}{A^2}\right\} \tag{4}$$

If $q = 0$, the spiral is unstable. This is nothing but modulational instability in the focusing NLSE. Also, the helix is unstable in the anti-rotating case $q < 0$. In the co-rotating case, the spiral is stable if

$$\frac{2q}{A^2} > \frac{k^4A^4}{1+k^2A^2}$$

In particular, co-rotating straight vortex $k = 0$ is stable. If $q < 0$, such vortex is unstable; this is an analog of well-known Crow instability of a pair of anti-parallel vortices.

In the anti-rotating case, the development of helix instability leads to gluing of both vortices in a finite time. This process can be treated as reconnection of vortices. The final stage of this reconnection can be described by self-similar solution of equation (1):

$$\Psi = (t_0 - t)^{1/2+i\nu(q)}\,\Phi\left(\frac{z}{\sqrt{t_0 - t}}\right) \tag{5}$$

Here $\nu(q)$ is a nonlinear eigenvalue to be found numerically.

In the co-rotating case, Equation (1) has a rich family of solitonic solutions, which stabilize at $z \to \infty$ to a stable spiral. Such solitonic solution can be sophisticated, in particulary, knotted. Their stability is not properly studied yet.