Coaxial axisymmetric vortex rings – 150 years after Helmholtz

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The talk addresses the Section 6, entitled ‘‘Circular vortex-filaments’’, of the seminal paper of Helmholtz [1]. Here Helmholtz studies a system of the coaxial circular axisymmetric vortex filaments (the vortex rings) in an incompressible inviscid unbounded fluid. In the talk I want to elucidate the following points:

1. To provide a detailed account of interesting but almost unknown comments made by S.A. Chaplygin in the Russian translation [2] with pointing out some minor errors (without changing the final qualitative results) in the formulae (8a) of [1] (not corrected in the well-known Tait's translations [3], and only briefly mentioned in [4]).

2. To provide quantitative illustrations to the qualitative statement of Helmholtz [3, p. 510]:
We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.
If they have equal radii and equal and opposite angular velocities, they will approach each other and widen one another; so that finally, when they are very near each other, their velocity of approach becomes smaller and smaller. If they are perfectly symmetrical, the velocity of fluid elements midway between them parallel to the axis is zero. Here, then, we might imagine a rigid plane to be inserted, which would not disturb the motion, and so obtain the case of a vortex-ring which encounters a fixed plane.

The motion of the system of N thin coaxial vortex rings in an inviscid unbounded fluid with the vortex ring i has the middle the radius R_i, the intensity Γ_i, the core with a small cross section of radius a_i ($n_i = a_i / R_i << 1$) and axial position Z_i in the cylindrical coordinate system (r, ϕ, z).is described by a system of ordinary differential equations of the first order, (the so-called Dyson's vortex ring model [5]):

$$\frac{dR_i}{dt} = -\frac{1}{\Gamma_i R_i}\frac{\partial U}{\partial Z_i}, \qquad \frac{dZ_i}{dt} = \frac{\Gamma_i}{4\pi R_i}\left(\ln\frac{8R_i}{a_i} - \frac{1}{4}\right) + \frac{1}{\Gamma_i R_i}\frac{\partial U}{\partial R_i}, \qquad a_i^2 R_i = const, \tag{1}$$

where
$$U = \sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{\Gamma_i\Gamma_j}{4\pi}\sqrt{R_i R_j}\left[\left(\frac{2}{k_{ij}} - k_{ij}\right)\mathbf{K}(k_{ij}) - \frac{2}{k_{ij}}\mathbf{E}(k_{ij})\right], \qquad k_{ij}^2 = 4R_i R_j / [(R_i + R_j)^2 + (Z_i - Z_j)^2],$$

with the initial conditions: $R_i(0) = R_i^0$, $Z_i(0) = Z_i^0$, $n_i(0) = n_i^0$. Here $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptical integrals of the first and second kind accordingly.

The system of equations (1) has two independent invariants

$$P = \sum_{i=1}^{N}\Gamma_i R_i^2 = \text{const}, \qquad W = \sum_{i=1}^{N}\frac{\Gamma_i^2 R_i}{4\pi}\left(\ln\frac{8R_i}{a_i} - \frac{7}{4}\right) + U = \text{const}, \tag{2}$$

which correspond to the impulse and kinetic energy conservation laws accordingly.

Figure 1 shows the interaction of two identical vortex rings. Arrows show directions of motion, filled circles indicate positions of vortices for equal time intervals. The ‘leap-frog’ motion of the vortex rings is obvious. Detail study shows that periodic interaction of rings with the same signs of intensities is also possible for unequal vortices (Fig. 2). Fig. 3-5 show various types of the vortex rings interaction with opposite signs of intensity.

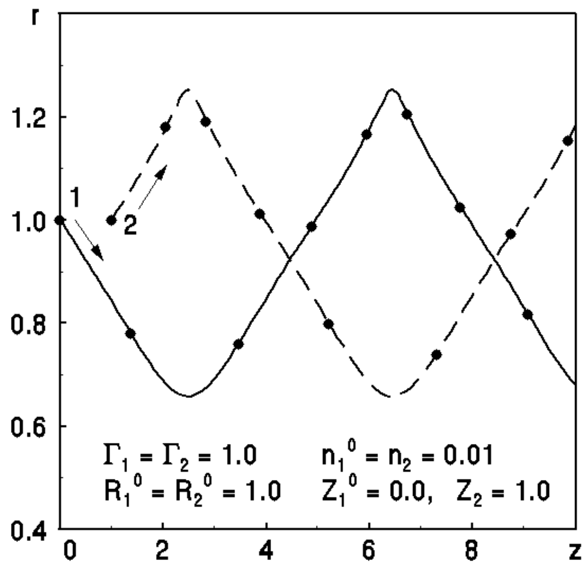

Fig.1. Periodic interaction of two equal vortex rings with same signs of intensity.

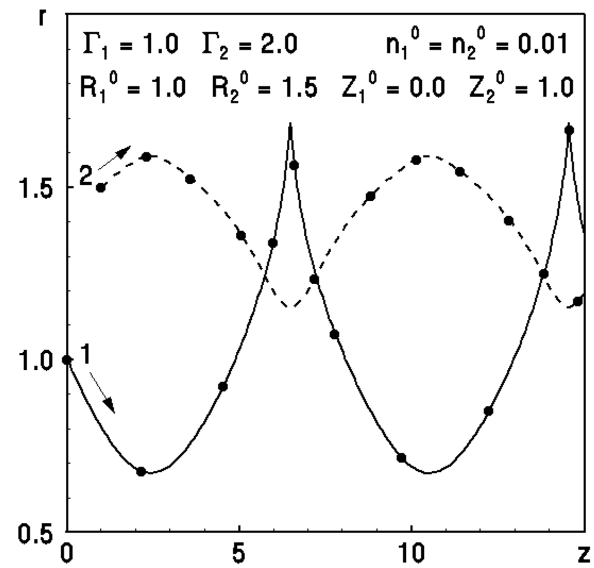

Fig.2. Periodic interaction of two nonequal vortex rings with same signs of intensity.

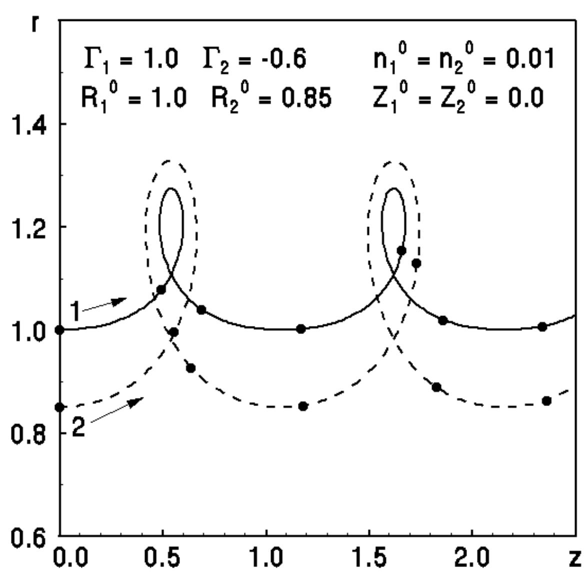

Fig.3. Periodic interaction of two nonequal vortex rings with opposite signs of intensity.

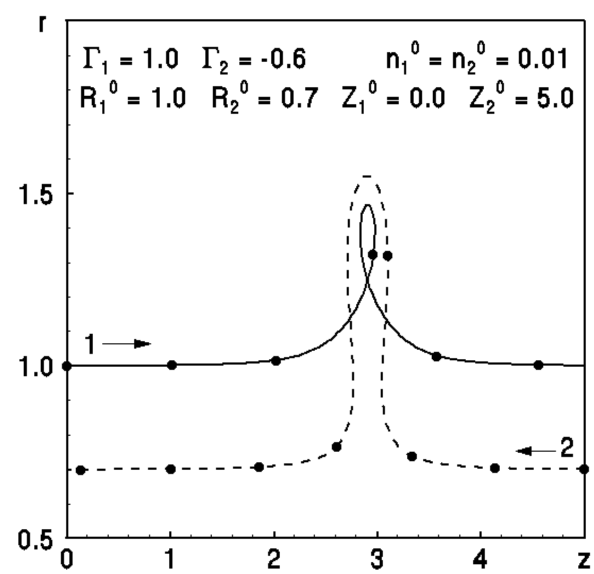

Fig.4. Periodic interaction of two nonequal vortex rings with opposite signs of intensity.

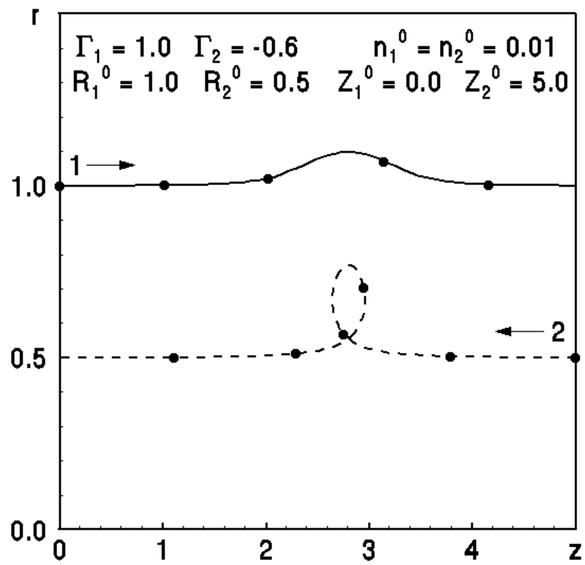

Fig.5. Periodic interaction of two nonequal vortex rings with opposite signs of intensity.

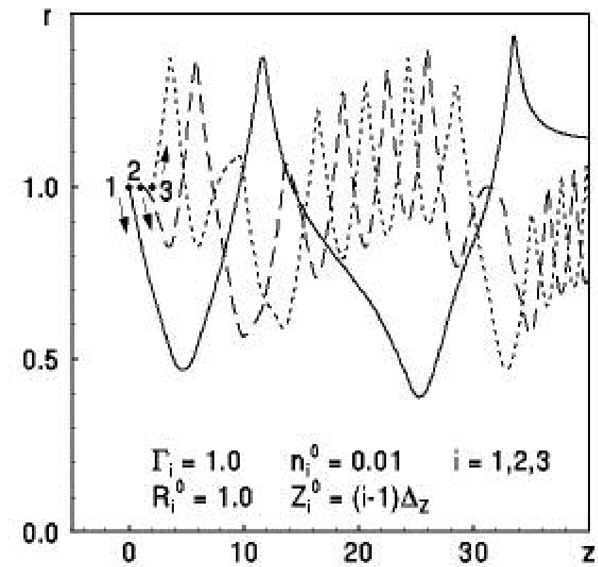

Fig.6. Interaction of three axisymmetrical vortex rings.

Increasing of number of vortex rings results in changes in the system (1), which becomes non-integrable. In this case vortex dynamics becomes more difficult, and small changes in the initial parameters lead to substantial changes in vortex trajectories. The typical example of interaction of three identical vortices is shown in Fig. 6. Researches shows that vortex rings have a typical tendency of forming the local double structures, where vortices periodically pass one through another.

3. To provide an overview of the results on the regular and chaotic motion of coaxial vortex rings and their interaction with solid bodies obtained in Kiev in the period of 1987-2007.

References

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