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CHAOTIC STREAMLINES IN THE FLOW OF KNOTTED AND UNKNOTTED VORTICES

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Summary This paper describes the motion and the flow produced by a thin vortex filament lying on the surface of a torus. The vortex filament progresses along and rotates about the symmetry axis of the torus in an almost steady manner, while approximately preserving its shape. Streamlines are analysed in a frame moving with the vortex and it is found that they are regular at small and large distances from the filament and chaotic at intermediate distances.

VORTEX EVOLUTION

In 1875 Kelvin [5] hypothesised that vortex filaments lying on the surface of a torus could exist as steady structures and that they would be stable "provided only that the core is sufficiently thin." The existence of such solutions to the Euler equations is still an open question, but in the so-called "localized induction approximation" such solutions are known to exist [1] and to be either stable or to have slowly growing instabilities [3].

For analytical and numerical convenience, we use as initial condition a vortex filament of strength Γ which is uniformly coiled on a torus of radius r_0 and cross-section πr_1^2. Therefore the filament is given, in Cartesian coordinates, as follows:

$$\begin{aligned} x &= (r_0 + r_1 \cos\phi)\cos\theta, \\ y &= (r_0 + r_1 \cos\phi)\sin\theta, \\ z &= r_1 \sin\phi. \end{aligned}$$

Here ϕ is the angle around the torus' centerline and θ is the angle around the torus' symmetry axis. They are given by $\phi = qs$ and $\theta = ps$, where p and q are integers and s is a parameter in the range $0 - 2\pi$. Hence, before closing on itself, the filament $T_{p,q}$ makes p revolutions around the torus' symmetry axis and q revolutions around the torus' centerline. These numbers determine the topology of the vortex filament, as follows: when $p > 1$, $q > 1$ (p, q co-prime integers) the filament is a toroidal knot, when either $p = 1$ or $q = 1$ the filament is a toroidal unknot. In the latter situation, however, it is useful to make a distinction between the cases $p = 1$ and $q > 1$ (toroidal helices) and $p > 1$ and $q = 1$ (toroidal loops).

We compute the induced velocities with the Rosenhead-Moore approximation to the Biot-Savart law [4], and move the filament forward in time with a fourth-order Runge-Kutta scheme with fixed time step. Since the filament approximately preserves its shape, the number of nodes used to represent the filament is kept constant throughout the simulations. All vortex filaments are observed to progress along the symmetry axis of the torus with an approximately uniform speed U, and to rotate around this line with an approximately uniform angular speed Ω (an

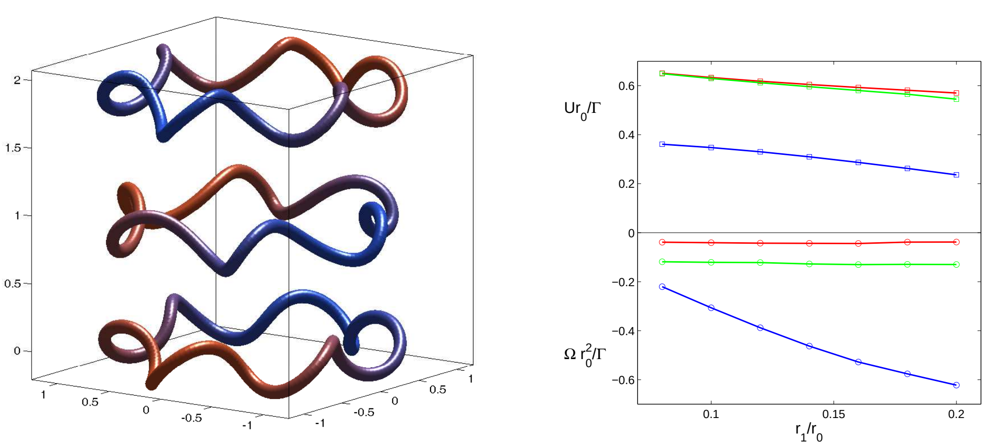

Figure 1. Drift U and rotation Ω of vortex filaments coiled on the surface of a torus. The left panel shows the evolution of a toroidal helix $T_{1,6}$ (the filament has volume to facilitate visualising the 3D shape and colour to mark position along the filament). The right panel shows how U (squares) and Ω (circles) depend on the aspect ratio of the torus (r_1/r_0) in three cases: a toroidal helix $T_{1,6}$ (blue), a toroidal loop $T_{2,1}$ (red), and a toroidal knot $T_{2,3}$ (green).

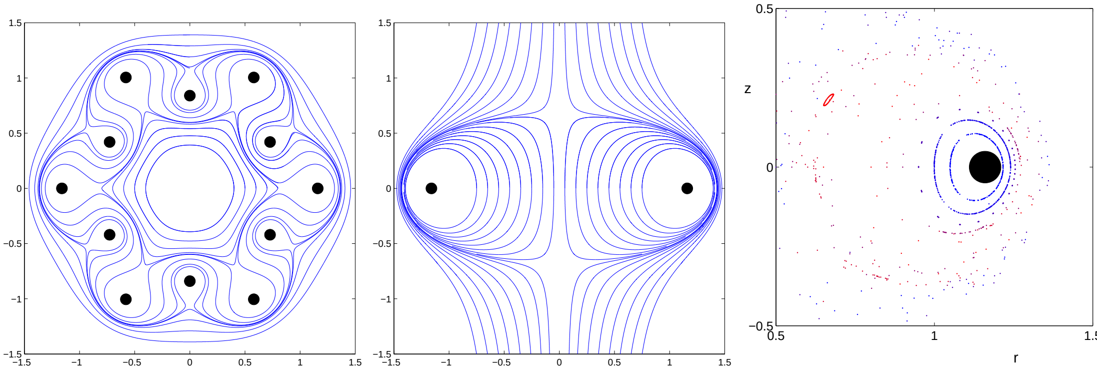

Figure 2. Left: "Streamlines" computed with the velocity component on a plane containing the torus' centerline. Middle: "Streamlines" computed with the velocity component on a meridional plane. Right panel: Poincaré section constructed plotting the intersections of actual streamlines with a meridional plane (the colour identifies a particular streamline) . The black circles represent the filament cross section.

example of a toroidal helical vortex is shown in the left panel of figure 1). U grows as the value of p increases, but is almost unaffected by the value of q. Similarly, the magnitude of Ω grows as q increases but is almost unaffected by p. As the aspect ratio of the torus (r_1/r_0) increases the magnitude of U decreases and that of Ω increases (see right panel of figure 1). The azimuthal speed that results from the rotation of toroidal helical vortices agrees reasonably well with the progressive speed of cylindrical helical vortices [2].

FLOW GEOMETRY

If the aspect ratio of the supporting torus (r_1/r_0) is small, the velocity field of a toroidal vortex filament can be considered as a perturbation of the velocity field of a circular vortex ring of radius r_0 and strength $p\Gamma$. We must thus first describe the flow geometry of the latter. A circular vortex ring moves with constant velocity U and, in a reference frame fixed on the ring, there is always a set of closed stream surfaces. The surface with the largest capacity is called separatrix, because the fluid contained within this surface is carried away by the vortex ring while the fluid outside is not. The capacity of the separatrix increases as U decreases. The shape also changes: below some critical value of U the separatrix is a flattened spheroid; above that value it is a toroid (i.e. it is topologically equivalent to a torus but its cross section is not a circle). In all cases, inside the separatrix there is an infinite set of closed stream surfaces with the shape of toroids.

Perturbing this axially-symmetric, steady flow is analogous to perturbing a two-dimensional, steady flow. Therefore the streamlines in the three-dimensional flow of the vortex filament $T_{p,q}$ should have some characteristics in common with particle trajectories in a time-periodic two-dimensional flow. This is indeed the case for the toroidal helical vortex $T_{1,6}$ shown in figure 2. The "streamlines" on a plane containing the torus' centerline show a general circulation around the symmetry axis (the sense, not indicated in the figure, is counter-clockwise), the "streamlines" on a meridional plane resemble the streamlines of a slow circular vortex ring (recall that the vortex filaments $T_{p,q}$ move at a lower speed than the equivalent circular vortex ring). Finally the Poincaré section shows that only streamlines close to the filament core lie on a toroid, whereas the scattered points farther from the core indicate the presence of chaotic streamlines.

A vortex filament $T_{p,q}$ evolves with only small changes in shape; yet we expect that this perturbation will further erode the closed stream surfaces and produce chaotic particle paths in larger flow regions. This is the subject of current research.

References

[1] Kida, S. 1981 A vortex filament moving without change of form. *J. Fluid Mech.* **112**, 397–409.

[2] Levy, H. and Forsdyke, A.G. 1928 The steady motion and stability of a helical vortex. *Proc. R. Soc. Lond. A* **120**, 670–690.

[3] Ricca, R.L., Samuels, D.C. & Barenghi, C.F. 1999 Evolution of vortex knots. *J. Fluid Mech.* **391**, 29–44

[4] Saffman, P.G. 1995 *Vortex Dynamics*. Cambridge University Press.

[5] Thomson, W. 1875 Vortex statics. *Proc. R. Soc. Edinb.* **9**, 59–73. Reprinted in *Mathematical and physical papers*, vol 4., 115–128.