VON KÁRMÁN STREETS ON THE SPHERE

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1. Abstract

A Von Kármán street (VKS) on the sphere consists of two symmetrically skewed closed rings, each containing N evenly spaced vortices. We discuss both single and double von Kármán streets on the sphere, with emphasis on their general solution, algebraic structure, statistical properties and bifurcations in the streamline topology. For both the single and double Von Kármán streets, the effect of adding vortices to the poles is studied. The system is posed as a problem in linear algebra, where the vector $\mathbf{\Gamma} \in \Re^M$, describes the strengths of a system containing M vortices, and is solved as a function of the configurations geometry, namely by the condition $A\mathbf{\Gamma} = 0$. A is non-square, and is referred to as the configuration matrix. The solution to $\mathbf{\Gamma}$ lies in the null space of A, which is found using singular value decomposition (SVD). In addition, the Shannon entropy is extracted from the singular value distribution and discussed for different configurations .

1.1. **The Single Von Kármán Street:** A single VKS contains one ring in the northern hemisphere, and a second in the southern hemisphere, with the two having N vortices and the same latitude ϕ from the poles. For any choice of N, except $N = 3$, the dimension of the solution of $\mathbf{\Gamma}$ is one, with the vortices of the northern ring having equal and opposite strength to those of the southern ring. For $N = 3$, the dimension of the solution of the system is 3, with one vortex in the northern ring being equal and opposite to the vortex longitudinally spaced by π in the southern ring. Adding pole vortices makes the dimension of the solution 3 for any N, but in order for the vortex street to have rings with equal and opposite strength, the strength of the pole vortices must be chosen to be $\Gamma_{NP} = -\Gamma_{SP} \in \Re$. A bifurcation study was performed by varying the strength the pole vortices, and tracking the change in the streamline topology. The stagnation points are key in describing the change in topology, and they move from the poles to the street vortices as the magnitude of the pole vortices strength increases.

1.2. **The Double Von Kármán Street:** A double VKS consists of one VKS in the northern hemisphere, and a second in the southern hemisphere. Using the SVD method, it was found that the dimension of the null space for this system is one, see Figure 1. The outermost rings have equal and opposite strength Γ, and are referred to as the ϕ_1-rings. The strength of the two (equal and opposite) inner rings, referred to as the ϕ_2-rings, is $\pm\alpha\Gamma$, where $\alpha \in \Re$. Since the vortex ring pairs do not, in general, have equal and opposite strengths ($\alpha \neq 1$), this system is not a classic double VKS. A double VKS ($\alpha = 1$) can be achieved by adding vortices at the north and south poles. The dimension of the null space for the system with poles is 3. The additional dimensionality of the solution comes from the pole vortices that have independent strengths Γ_{NP} and Γ_{SP}. Since the pole vortices affect the strength of the street vortices, the poles can be chosen so to make the constant α of Figure Figure 1 to be $\alpha = 1$, thus achieving the double VKS shown in Figure 1. The pole strengths must be equal and opposite with $\Gamma_{NP} = -\Gamma_{SP} = \beta\Gamma$, where the parameter

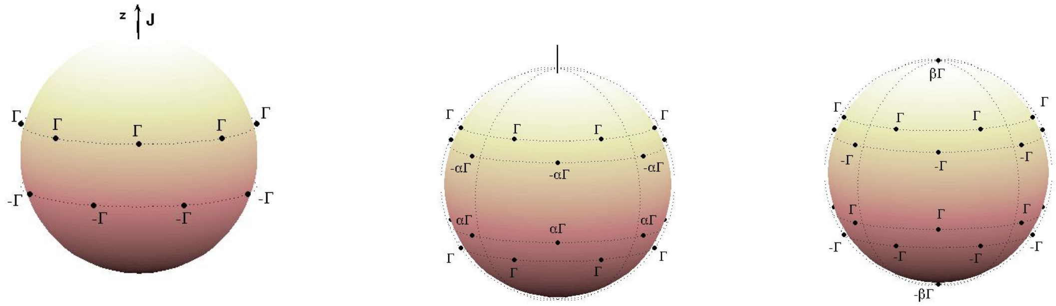

FIGURE 1. Schematic diagrams of the single and double Von Kármán Streets on the sphere, with an illustration of the general vortex solutions and vorticity vector **J**.

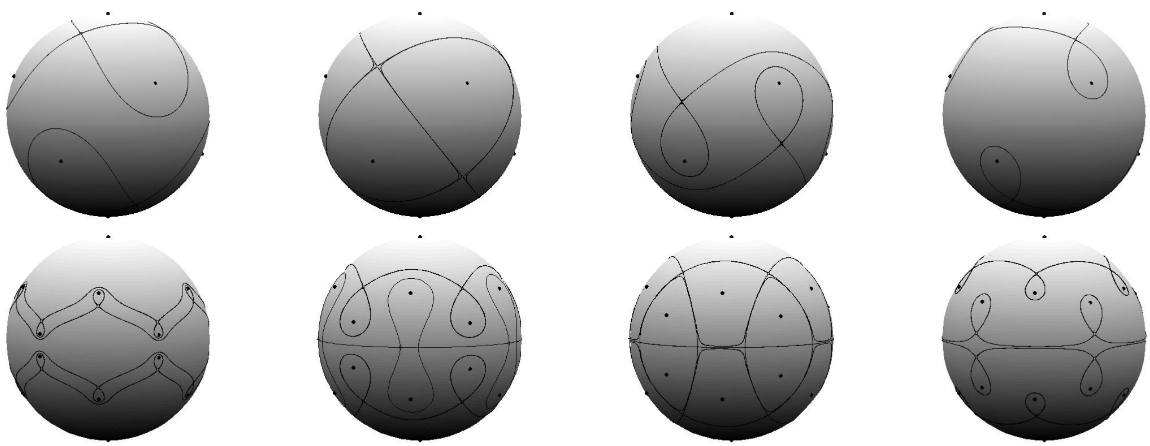

FIGURE 2. Examples of different streamline topologies found in the Single Von Kármán Street (upper row) and Double Von Kármán Street (lower row).

$\beta = \beta(\phi_1, \phi_2, N) \in \Re$ is a function of the configuration and the solution collapses into a one-dimensional solution of the form $\mathbf{\Gamma} = \Gamma[1, ..., 1, -\alpha, ..., -\alpha, \alpha, ..., \alpha, -1, ... -1, \beta, -\beta]^T$, where $\mathbf{\Gamma}$ has length $4N + 2$ with the ring vortices stacked from north to south and pole vortices placed at the end of the state representation vector.

A bifurcation study was performed on the streamline topology by keeping ϕ_1 constant while varying the ratio ϕ_1/ϕ_2. The stagnation points follow a more complex path than in the case of a single von Kármán street. They travel along the longitudes of the vortex streets, and on the equator. The topology changes when either (1) stagnation points merge or split, or when (2) separatrices merge or split.