Self-Propulsion of a Free Hydrofoil with Localized Discrete Vortex Shedding: Analytical Modeling and Simulation

Scott David Kelly [1] and Hailong Xiong [2]

Abstract

We present a model for the self-propulsion of a deforming hydrofoil in a planar ideal fluid. We begin with the equations of motion for a deforming foil interacting with a system of point vortices and demonstrate that these equations possess a Hamiltonian structure. We add a mechanism by which new vortices can be added to the fluid near the trailing edge of the foil according to a time-periodic Kutta condition, imparting thrust to the foil such that the total impulse in the system is conserved. Simulation of the resulting equations reveals at least qualitative agreement with the observed dynamics of fishlike locomotion.

Introduction. Recent work by the authors and others has addressed two complementary problems in fluid-body interactions from the standpoint of analytical mechanics. The self-propulsion of a deformable body in an ideal fluid devoid of vorticity is treated as a problem in Lagrangian mechanics in [KM96, Kel98, KMRMH05], with the simplistic addition of liftlike forces due to circulation described in [KH06]. Hamiltonian models for the interactions of free rigid bodies with discrete vortex structures are presented, meanwhile, in [SMBK02, BMR03, SSKM08]. In the present paper, we merge these lines of research to provide a model for the self-propulsion of a deformable body in a planar ideal fluid — specifically, a hydrofoil defined by a time-varying conformal map — which is able to shed vorticity discretely from a single point on its surface in accordance with a periodically applied Kutta condition. The shedding of each vortex is accompanied by the application of an impulsive force to the hydrofoil in order to conserve the total impulse in the system. Between vortex shedding events, the equations of motion possess a Hamiltonian structure which extends that underpinning the interaction of a free rigid body with a system of vortices. Computational experiments with this model demonstrate its qualitative fidelity to the observed dynamics of a self-propelled fishlike robot in the first author's lab.

Foil Shapes and Complex Potentials. We model the contour of a hydrofoil with time-varying shape as the image of a circle in the complex plane under a conformal map $z = x + iy = F(\zeta)$ with time-varying parameters s_j. We require the area within the foil to remain constant in time to avoid an infinite term in the kinetic energy of the resulting fluid-foil system. We express the dynamics of the moving foil relative to the foil-fixed z-frame. In between vortex shedding events, the flow resulting from the motion of the foil and vortices — assuming the fluid to be at rest infinitely far away, and excluding small domains around the the vortices themselves — is determined by a potential function of the form

$$W(z) = w(\zeta) = Uw_1(\zeta) + Vw_2(\zeta) + \Omega w_3(\zeta) + \sum_j \dot{s}_j w_{s_j}(\zeta) + \sum_k w_{v_k}(\zeta),$$

where U and V are the x- and y-components of the foil's translational velocity, Ω is its angular velocity, and the complex potentials w_{v_k} represent the contributions of the vortices to the flow.

Conservation of Impulse Between Vortex Shedding Events. Relative to the foil-fixed frame, the total linear and angular impulse in the fluid-foil system may be expressed in the form

$$\begin{bmatrix} L_x \\ L_y \\ P \end{bmatrix} = I(\mathbf{s}) \begin{bmatrix} U \\ V \\ \Omega \end{bmatrix} + B(\mathbf{s})\dot{\mathbf{s}} + \sum_k \gamma_k K_k(\mathbf{s}, z_k),$$

where $\mathbf{s}$ is the vector of shape parameters for the foil, γ_k and z_k are the strength and location of the kth vortex, $I(\mathbf{s})$ and $B(\mathbf{s})$ are matrices with appropriate dimensions, and the K_k are vectors. The motion of the foil is governed by *Kirchhoff's equations*

$$\left(\frac{d}{dt} + \bar{\Omega}\times\right)\mathbf{L} = 0, \qquad \frac{d\mathbf{P}}{dt} + \mathbf{U}\times\mathbf{L} = 0,$$

where $\bar{\Omega} = [0\,0\,\Omega]^T$, $\mathbf{L} = [L_y\,L_x\,0]^T$, $\mathbf{P} = [0\,0\,P]^T$, and $\mathbf{U} = [U\,V\,0]^T$. The motion of the kth vortex is determined using *Routh's rule* [Saf92] such that

$$\dot{\zeta}_k = \left(\overline{\frac{dW_k}{dz}} - (U + iV + i\Omega z_k) - \sum_j \frac{\partial F}{\partial s_j}\dot{s}_j\right)\frac{1}{F'(\zeta_k)},$$

where $W_k(z) = W(z) - i\gamma_k \log(z - z_k)$.

[1] Department of Mechanical Engineering and Engineering Science, University of North Carolina at Charlotte, scott@kellyfish.net
[2] Quantitative Risk Management Inc., xiong@snapturn.com

Hamiltonian Structure. Observing that

$$I(\mathbf{s})\begin{bmatrix}U\\V\\\Omega\end{bmatrix}=\begin{bmatrix}L_x\\L_y\\P\end{bmatrix}-B(\mathbf{s})\dot{\mathbf{s}}-\begin{bmatrix}\int \mathbf{l}\times(\mathbf{n}\times\nabla\phi_v)ds\\-\frac{1}{2}\int l^2(\mathbf{n}\times\nabla\phi_v)ds\end{bmatrix}-\sum_k(-2\pi\gamma_k)\begin{bmatrix}y_k\\-x_k\\-\frac{1}{2}(x_k^2+y_k^2)\end{bmatrix}$$

in the notation of [SMBK02], we may define the Hamiltonian function

$$H(L_x,L_y,P,x_1,y_1,\ldots,x_N,y_N)=\frac{1}{2}[U\ V\ \Omega]I(\mathbf{s})\begin{bmatrix}U\\V\\\Omega\end{bmatrix}-2\pi H_1$$

when N wake vortices are present, where

$$H_1=\sum_k\gamma_k\sum_j\dot{s}_j\psi_{s_j}(x_k,y_k)-\frac{1}{2}\sum_k\gamma_k^2\left(\log\left|\zeta_k\overline{\zeta_k}-r_c^2\right|+\log\left|F'(\zeta_k)\right|\right)+\frac{1}{2}\sum_k\sum_{j\neq k}\gamma_k\gamma_j\left(\log\left|\zeta_k-\zeta_j\right|-\log\left|\zeta_k\overline{\zeta_j}-r_c^2\right|\right).$$

If $\mu=[L_x\,L_y\,P]^T$, then the motion of the foil is governed in between vortex shedding events by the *Lie-Poisson equations*

$$\dot{\mu}=\mathrm{ad}^*_{\delta H/\delta\mu}\mu$$

on $\mathfrak{se}^*(2)$, while the positions of the vortices evolve such that

$$(-2\pi\gamma_k)\frac{dx_k}{dt}=\frac{\partial H}{\partial y_k},\qquad(-2\pi\gamma_k)\frac{dy_k}{dt}=-\frac{\partial H}{\partial x_k}.$$

Vortex Shedding. We introduce new vortices to the wake near the trailing edge of the foil in simultaneous accord with the conservation of impulse and the Kutta condition. This leaves flexibility in the way in which we select the (coupled) position and strength of each new vortex. We provide a comparison of different methods for doing so in [XK], and settle on a method described in [ST95] for the present discussion. According to this method, we situate each new vortex along a contour interpolated from the trailing edge of the foil to the current position of the last vortex shed, and determine the strength of each new vortex as a function of its initial position. Vortices shed at different times may have different strengths; the strength of each shed vortex is assumed to remain constant thereafter. Enforcing the constraints described above, we introduce each vortex with strength γ_k at the image of the point ζ_k such that

$$\left.\frac{dw}{d\zeta}\right|_{\zeta=\zeta_T}=0,$$

where ζ_T is the preimage of the trailing edge of the foil under the transformation defining the foil's shape, and such that

$$I(\mathbf{s})\begin{bmatrix}\Delta U\\\Delta V\\\Delta\Omega\end{bmatrix}+\gamma_k K_k(\mathbf{s},z_k)=0,$$

where Δ refers to the impulsive change in the foil's velocity as a result of shedding.

Simulation Results. Figure 1 shows results from two different simulations based on our model. The top row portrays the acceleration from rest of a von Mises foil undulating according to periodic variations in two shape parameters. Wake vortices are depicted in the snapshot on the left as tiny colored circles; red circles correspond to clockwise vortices and blue circles to counterclockwise vortices. We observe the roll-up of wake vorticity into staggered coherent structures; over time these assume the form of an inverse Kármán vortex street, consistent with experimental observations of the wakes trailing oscillating foils [TT95]. The plot on the right compares the predicted x- and y-displacement of the foil-fixed frame over time to that which would be measured were the mechanism for vortex shedding disabled ("*ns*"), underscoring the role played by vortex shedding in thrust development. The bottom row depicts the execution of a *snap turn* by a Joukowski foil, the camber of which is varied rapidly about zero through a single sinusoidal period, providing sufficient momentum for the foil to coast in an oblique direction thereafter. In both cases, the model predicts behavior which is at least qualitatively consistent with that of a fishlike robot in the first author's lab. The authors are currently undertaking to validate the model more carefully using this robotic system.

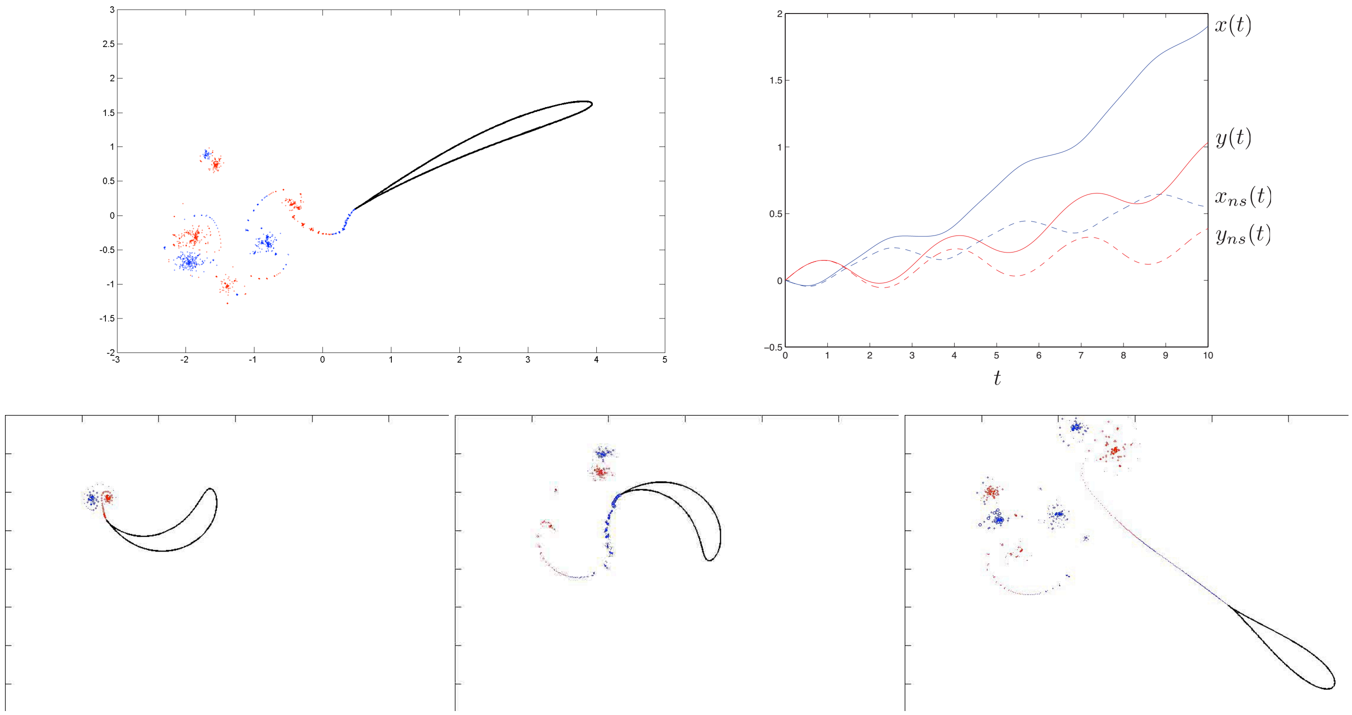

Figure 1: Simulation results. *Top row*: Forward acceleration from rest by an undulating von Mises foil shedding vortices. *Bottom row*: Snap turn by a Joukowski foil executing aggressive changes in camber.

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