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Turbulence transition in pipe flow: some open questions

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Abstract
The transition to turbulence in pipe flow is a longstanding problem in fluid dynamics. In contrast to many other transitions it is not connected with linear instabilities of the laminar profile and hence follows a different route. Experimental and numerical studies within the last few years have revealed many unexpected connections to the nonlinear dynamics of strange saddles and have considerably improved our understanding of this transition. The text summarizes some of these insights and points to some outstanding problems in areas where valuable contributions from nonlinear dynamics can be expected.

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1. Introduction

The equations of fluid flow come naturally with a built in nonlinearity in the form of the convective derivative and, hence, constitute a popular playground for applications of nonlinear dynamical systems theory. This mutually beneficial relationship has figured prominently in bifurcation theory, the routes to chaos and the development of nonlinear dynamics and fluid mechanics in general, see, e.g., Chandrasekhar (1961), Drazin and Reid (1981) and Koschmieder (1993). It is therefore appropriate to commemorate the 20th anniversary of *Nonlinearity* by highlighting recent developments and open problems in an area that also has a reason to celebrate an anniversary: 2008 marks the 125th anniversary of Osborne Reynolds's seminal papers (Reynolds 1883a, b) on the 'conditions which determine whether the flow of a fluid is sinuous' in which he describes his observations on the intermittent transition to turbulence in circular pipes (see Jackson and Launder (2007) for historical background on Reynolds' publications). Despite its long tradition and obvious practical relevance in many engineering situations, there are many aspects of this transition that have puzzled scientists. For instance, one might expect that after more than a century the 'critical flow rate', measured by the dimensionless Reynolds number *Re*, for the transition to turbulence in pipe flow should be firmly established. Instead, one finds in the literature Reynolds numbers which range between 1000 (Prandtl and Tietjens 1931) and more than 3000. It turns out that this wide range is a natural consequence of the intrinsic properties of the system, and directly linked to

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the presence of a chaotic saddle. In the following, I will summarize the work that has led to this observation as well as several other key developments, and describe some open questions that have emerged out of these investigations. More background information as well as more details may be found in the recent reviews (Kerswell 2005, Eckhardt *et al* 2007) and the proceedings (Mullin and Kerswell 2005).

The outline is as follows. In section 2 I will summarize a few experimental and theoretical facts about pipe flow. Section 3 deals with coherent structures and section 4 with their connections in state space. Section 5 discusses the issues connected with the observed transience of turbulence. In section 6 we focus on the edge of chaos and in section 7 on the minimal perturbations needed to trigger turbulence. The global dynamics in relation to the localization of the turbulence in puffs is discussed in section 8. Finally, in section 9 we briefly outline connections to other shear flows.

2. Observations and elementary properties

Pressure driven flow down a smooth circular pipe develops a parabolic velocity profile sufficiently far from the inlet. In the usual dimensionless units one measures length in units of the diameter D and velocities in units of the mean velocity U. From these and the viscosity of the fluid ν one can form a dimensionless number, the Reynolds number $Re = UD/\nu$. Hydrodynamic similarity theory states that all flows with the same Reynolds number, independent of flow speed or diameter of the pipe, behave in the same manner.

The unusual properties of the transition to turbulence in pipe flow are causally connected to the fact that the parabolic profile is linearly stable against infinitesimal perturbations (Salwen *et al* 1980, Brosa 1986, Meseguer and Trefethen 2003). Experimentally, the laminar flow has been maintained for Reynolds numbers as high as 100 000 (Pfenniger 1961). However, the operator obtained after linearizing the Navier–Stokes equation around the laminar profile is not normal, and perturbations can temporarily extract energy from the laminar flow and be amplified, before finally decaying (Boberg and Brosa 1988, Reddy *et al* 1993, Trefethen *et al* 1993, Waleffe 1995, Henningson 1996, Schmid and Henningson 1999, Grossmann 2000, Jachens *et al* 2006, Kim and Moehlis 2006). This has led (Meseguer and Trefethen 2003) to suggest that for Reynolds numbers in excess of 10^7 the transient amplification is so strong that both experimentally and numerically it becomes practically impossible to control the perturbations and to prevent the transition.

In view of the linear stability, the transition to turbulence requires finite amplitude perturbations. These can derive from perturbations present in the reservoir or triggered in the inflow region and then swept into the pipe. A typical experiment shows an intermittent variation between laminar and turbulent domains which move downstream (for experimental demonstrations on the original apparatus used by Reynolds, see the images provided by Homsy *et al* (2004) or Eckhardt *et al* (2007); for time traces, see Rotta (1956)). For controlled experiments one turns to controlled perturbations, such as pressure pulses from loudspeakers (Wygnanski *et al* 1975), jets of fluids injected into the pipe (Darbyshire and Mullin 1995) or devices such as the iris diaphragm (Durst and Ünsal 2006) that temporarily blocks the flow. For sufficiently low Reynolds numbers these perturbations decay as they are swept downstream and do not recover. For Reynolds numbers up to about 2700, the perturbations develop into localized patches of about $30D$ lengths which move downstream with a speed close to but not identical to the mean velocity: this implies that there is a continuous flux of liquid through the patch. Interestingly, these patches keep their length. For higher Reynolds number, the upstream and downstream fronts move with different speeds and the localized patches spread out along the pipe axis. These structures are called puffs and slugs, respectively, and are discussed extensively in Wygnanski and Champagne (1973), Wygnanski *et al* (1975). Almost all the discussions here concern Reynolds numbers below about 3000 and the dynamics in puffs.

From a mathematical perspective, the problem is the characterization of an initial value problem for a non-linear, non-local partial differential equation, the Navier–Stokes equation. The temporal evolution of a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ obeys

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} \tag{1}$$

together with the incompressibility condition

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

and appropriate boundary conditions. Taking the divergence of the Navier–Stokes equation (1) gives a Poisson equation for the pressure,

$$\Delta p = -\nabla \cdot ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}), \tag{3}$$

which results in a non-local dependence on the velocity gradients. The boundary conditions are that the fluid velocity vanishes at the walls. In the axial direction periodic boundary conditions are often used. A first question, worthy of a million USD in bounty, concerns the smoothness of solutions starting from smooth initial conditions for all times (see Feffermann (2000) for the prize question and Doering and Gibbon (1995) for some background information). Should it be possible to arrive at singularities in finite times, then numerical representations on finite-dimensional truncations of spaces of basis function become of dubious quality. In the absence of any positive evidence for singularities we will assume that the numerical representations are acceptable, modulo the usual issue of sufficient resolution of the fine scales.

3. Lowest Reynolds number for coherent structures

A persistent turbulent dynamics requires in its state space the presence of persistent structures other than the laminar profile. While there are examples of dynamical systems without periodic orbits, the most likely candidates for such persistent structures are some forms of periodic motions. Perhaps the simplest form is travelling waves, where a certain velocity field moves downstream without changing its form, $\boldsymbol{u}_{\mathrm{TW}}(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x} - c\boldsymbol{e}_z t)$. More complicated ones have a periodic or quasi-periodic time-dependence and or come in the form of helical waves where a translation in time and a translation in the downstream or azimuthal direction are coupled (so-called relative periodic states). The method of choice for converging such states is the Newton method, combined with various methods for obtaining good initial conditions, see Faisst and Eckhardt (2003) and Wedin and Kerswell (2004) for pipe flow and Nagata (1990), Clever and Busse (1997), Waleffe (1998, 2001, 2003) and Viswanath (2007b) for other flows.

For pipe flow, the nontrivial states that were identified first are families of coherent structures with symmetric arrangements of vortices (Faisst and Eckhardt 2003, Wedin and Kerswell 2004) (figure 1). More recently, asymmetric states, still of the travelling wave type, have been identified (Pringle and Kerswell 2007). Secondary bifurcations of the Hopf type can then lead to the creation of periodic orbits. The critical Reynolds number at which the first symmetric structures appear is around 1250, that for the asymmetric states around 770. The question that derives from this observation is

Question 1. Are there any persistent coherent states, of travelling wave or more complicated types, with Reynolds numbers below 770?

That the lowest coherent states can be more complicated than fixed points or travelling waves (which are essentially fixed points in a comoving frame of reference) can be seen in studies of low-dimensional dynamical system for shear flows, where the states that extend to the lowest Reynolds numbers are indeed periodic ones, with fixed points appearing at much higher Reynolds numbers only (Moehlis *et al* 2004, 2005). On the other hand, this may be a resolution effect, since the low-dimensional model is most closely related to plane Couette flow, and there the lowest lying states are, as far as we know, again of the fixed point type (Nagata 1990, Clever and Busse 1997, Waleffe 2003).

We do know from the study of the energy balance that for Reynolds numbers below about 80, all perturbations decay monotonically in energy (Joseph 1975). This then provides a lower bound. While it may be difficult to prove the existence of states, it might be possible to prove the absence of any by establishing an asymptotic decay for higher Reynolds numbers, following ideas from control theory (Hinrichsen *et al* 2004). The inverse question then is

Question 2. What is the maximal Reynolds number below which no coherent states can exist?

As a warm-up to this problem, it might be useful to address the related question in low-dimensional models (Eckhardt and Mersmann 1999, Moehlis *et al* 2004), where the problem is one of ordinary differential equations with quadratic nonlinearities, and where perhaps methods like Groebner bases can be put to good use.

4. Restructuring state space

Independent of whether the first coherent states appear at a Reynolds number of 770 or lower, it is an intriguing fact that experimental and numerical observations fail to come up with a somewhat longer lived turbulent dynamics for Reynolds numbers below about 1700 (Darbyshire and Mullin 1995, Hof 2004, Mullin and Peixinho 2006, Peixinho and Mullin 2006). Thus, while all the prerequisites for turbulence exist, the flow fails to show turbulence in any substantial manner. This suggests the following question:

Question 3. What happens in state space between the appearance of the first coherent structures and the experimental observation of turbulence?

The problem could be a qualitative one, in that the first states that appear are isolated and not sufficiently connected to maintain turbulent dynamics, so that observing turbulence requires a global bifurcation which then links different structures to form a noticeable attractor. The problem could also be simply a quantitative one, in that the structures form soon after the first bifurcations, and in a localized region in state space, and then simply grow and spread until they are sufficiently volume filling to be noticeable. Most likely, the answer to the problems involves a combination of both. One approach to addressing this problem requires the tracking of stable and unstable manifolds of coherent structures, as done for plane Couette flow by Gibson *et al* (2007).

An aspect of the restructuring is the observation that all states found in pipe flow so far are unstable, which means that they cannot be observed as permanent structures. However, it is possible to observe these states transiently in experiment and numerics (Hof *et al* 2004, Kerswell and Tutty 2007, Schneider *et al* 2007, Eckhardt *et al* 2008). The theory of chaotic systems holds that the frequency with which these states or more complicated ones appear is related to their instability (Cvitanovic and Eckhardt 1991, Eckhardt and Ott 1994).

Question 4. Can the frequency with which travelling waves or other coherent structures appear be related to their stability?

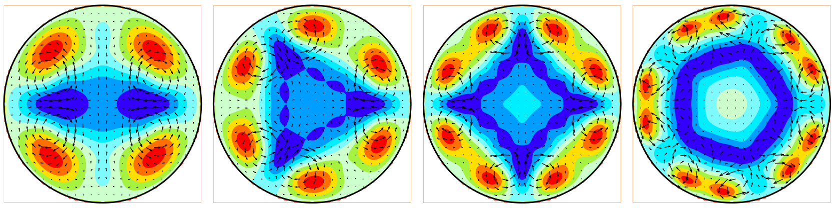

Figure 1. Examples of coherent structures in pipe flow. The profiles are obtained by averaging along the axis in order to highlight the symmetry. The arrows indicate the velocity in the cross section and the colour codes the downstream velocity relative to the parabolic profile. In the red regions the flow is faster than the parabola with the same mean flux, in the blue regions it is slower. From Faisst and Eckhardt (2003).

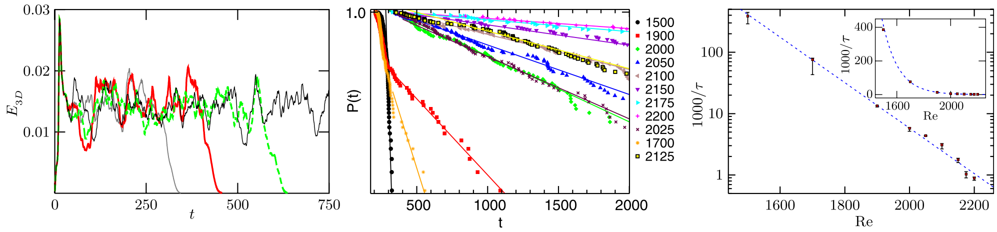

Figure 2. The energy traces in a turbulent section as a function of time show turbulent dynamics followed by a decay back to the laminar profile. When collected over many initial conditions, exponential distributions of lifetimes result (middle). The characteristic lifetimes increase rapidly with Reynolds number (right).

The studies (Kerswell and Tutty 2007, Schneider *et al* 2007) are a first step in this direction, fall short of establishing the quantitative relations between the frequency with which they appear in the observed signals and the stability of the identified objects.

5. Transient turbulence

Very often, turbulent dynamics is associated with the formation of an attractor: then the dynamics would be persistent, chaotic, with an invariant measure and many other nice features. Long ago, Brosa (1989) noticed that all his numerical runs returned to the laminar profile when followed long enough, and took the bold step to conclude that this was not specific to his numerics but an intrinsic property of turbulence: he suggested that all turbulence in pipe flow was transient (figure 2(*a*) has a modern version of this observation).

Interestingly, all numerical and experimental studies so far (Faisst and Eckhardt 2004, Mullin and Peixinho 2005, Hof *et al* 2006, Mullin and Peixinho 2006, Peixinho and Mullin 2006, 2007, Wilis and Kerswell 2007) agree that the lifetimes of turbulent pipe flow are exponentially distributed, so that the probability of still being turbulent after a time t is $P(t) \sim \exp(-t/\tau)$ for large t, with a characteristic time $\tau(Re)$ (figure 2(*b*)). The similarity sign indicates that this is only true for long times, and that certain short time transients have to be excluded (Hof *et al* 2007, Schneider 2007). Such behaviour is consistent with that expected for a chaotic saddle (Kadanoff and Tang 1984, Kantz and Grassberger 1985, Tél 1991).

The variation of the characteristic time with Reynolds number is an important and much debated subject. A transition from a chaotic saddle to a turbulent attractor would require some form of boundary crisis (Grebogi *et al* 1983) at which point $\tau(Re)$ diverges. Some measurements and numerical simulations are in agreement with this expectation (Faisst and Eckhardt 2004, Mullin and Peixinho 2006, Peixinho and Mullin 2006, Willis and Kerswell 2007). Other studies (Hof *et al* 2006, Schneider 2007) suggest that $\tau(Re)$ increases rapidly and present evidence that $\tau(Re)$ increases exponentially (figure 2(c)). These latter studies are based on several different experimental setups and are consistent with numerical studies for long pipes and long observation times (Schneider 2007). The very fact that there can be a dispute about the variations of lifetimes with Reynolds number shows that the presence (or absence) of an attractor in pipe flow at sufficiently high Reynolds numbers cannot be answered as easily as one might have expected.

Question 5. Is turbulence in pipe flow transient for all Reynolds numbers or is there a crisis bifurcation to an attractor? If there is no crisis, how rapidly does the lifetime increase with Reynolds number?

More computer power and additional experimental efforts will most likely reduce the statistical uncertainty and provide better data for the Reynolds number dependence $\tau(Re)$. In particular, they might indicate a different dependence on Re, just as other dynamical systems show a variety of variations of lifetimes with parameters (see, e.g., Crutchfield and Kaneko (1988), Kaneda (1990), Lai and Winslow (1995), Braun and Feudel (1996), Goren *et al* (1998), Rempel and Chian (2003)). It would also be helpful to find other indicators for the presence or absence of the boundary crisis, for instance in the dynamics of the edge state (see below), or in the fluctuations of the energy or dissipation rate. However, a definite answer to the appearance of an attractor requires constructive criteria for the identification of the necessary global bifurcation. But even then the turbulence need not be persistent, since spontaneous, noise-induced transitions between the two coexisting attractors could cause a relaminarization (Lagha and Manneville 2007, Schoepe 2004).

6. Edge of chaos

In the case of two coexisting attractors, there are basins and boundaries which separate the basins. If the turbulence is only transient, the turbulent dynamics connects to the laminar one, and there can be no basin boundary dividing state space into a turbulent and a laminar region. Hence, the concept of the dividing surface between the two regions has to be reconsidered and suitably generalized (Schneider and Eckhardt 2006, Skufca *et al* 2006, Schneider *et al* 2007, Vollmer *et al* 2007). Starting from the properties of the saddle state in a saddle-node bifurcation, one can use the lifetimes of perturbations, i.e. the time it takes to relax to the laminar profile, as an indicator: approaching the stable manifold from the laminar side of the saddle, the lifetime increases, reaches infinity and stays there, if the state on the other side is attracting. If the state on the other side is transient, there will be wild variations in lifetimes from one initial condition to a nearby one (Schmiegel and Eckhardt 1997, Faisst and Eckhardt 2004). Therefore, we denoted this point the 'edge of chaos' (Skufca *et al* 2006). All edge points seem to be connected in state space.

A second observation concerns the connections between edge points and the dynamics in the edge of chaos: numerical studies show an evolution towards a relative attractor, see Skufca *et al* (2006) for a model study, Schneider and Eckhardt (2006) and Schneider *et al* (2007) for pipe flow and Itano and Toh (2001), Toh and Itano (2003) and Viswanath (2007a) for other flows. A two-dimensional map can be used to characterize some of these properties, including possible bifurcations in the edge state and the appearance of chaos (Vollmer *et al* 2007). Most intriguingly, there is a possibility that this boundary can be fractal or not, depending on the ratio of two Lyapunov exponents.

Question 6. What are the properties of the edge of chaos and the invariant state within the edge? Is the edge of chaos a global relative attractor or are there additional attractors in the edge of chaos?

The models in Vollmer *et al* (2007) can easily be extended to cover multiple attractors in the edge, as observed in the model studied in Skufca *et al* (2006), but it would be nice to have an example in a realistic flow.

7. Minimal perturbations

In cases in which the laminar profile is stable, a finite perturbation is required to trigger turbulence. Experiments and the observations on non-normal amplification of perturbations show that the flow becomes increasingly sensitive to perturbations as the Reynolds number increases (Hof *et al* 2003). This suggests that the diameter of the basin of attraction of the laminar profile decreases with increasing Reynolds number. While there are results that suggest that the boundary contracts like $1/Re$ ((Hof *et al* 2003, Mellibovsky and Meseguer 2007) and, for low Re, figure 3), some experiments and numerics find steeper decays (Mellibovsky and Meseguer 2006, Peixinho and Mullin 2007, Philip *et al* 2007).

Question 7. How does the amplitude of the minimal perturbation required to trigger turbulence decay with Reynolds number?

To approach this question, one may look for minimal variations in the laminar profile which turn it unstable (Gavarini *et al* 2004), or for optimal 3d perturbations which trigger turbulence: (Ben-Dov and Cohen 2007) find a global minimum in energy norm for triggering secondary instabilities. It is also possible to study the scaling of the invariant state in the edge or perhaps of other relevant coherent states with Reynolds number. This is facilitated by the observation that the states do not become more complex with Re (Schneider 2007, Wang *et al* 2007). In simple models they can dominate the size of the basins (Eckhardt and Lathrop 2006), but here there is evidence that they maintain a finite distance from the laminar profile as $Re \to \infty$ (Wang *et al* 2007):

Question 8. Can one characterize the $Re \to \infty$ limit of coherent structures? Do the solutions approach the laminar profile or do some of them keep a finite distance for large Re?

If the states keep a finite distance from the laminar profile then the reduction in the basin of attraction of the laminar profile has to come from a scaling of the stable manifolds. Such a connection would also fit with the observations of kinks and folds in the edge (figure 3), which are typical for stable manifolds.

8. Turbulent spot dynamics

The coherent structures described earlier provide a convenient means for describing the dynamics in a finite section of the pipe with periodic boundary conditions. This, however, covers only one aspect of the dynamics, in that the experimentally induced turbulence is usually a localized one, confined to a domain of about $30D$ in length (Wygnanski and Champagne 1973, Wygnanski *et al* 1975) (see figure 4). If a shorter turbulent region is induced, it grows until it reaches this length; if a longer one is induced it either shrinks

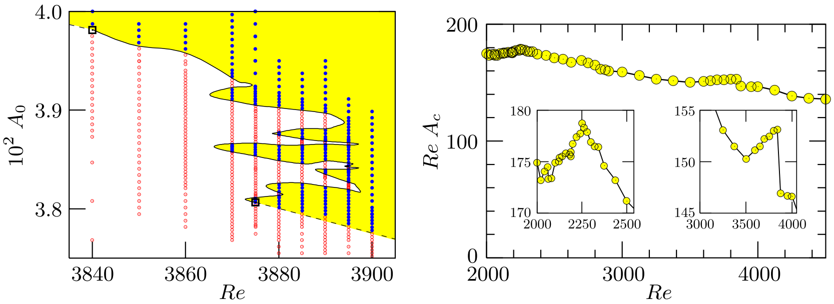

Figure 3. The boundary between laminar and turbulent (left) for a specific perturbation and the scaling of the boundary with Reynolds number. The insets on the right show modulations in the scaling curve connected with the folds on the left. The ordinate gives the amplitude of the perturbation (left) and the amplitude renormalized by the Reynolds number (right). From Schneider *et al* (2007).

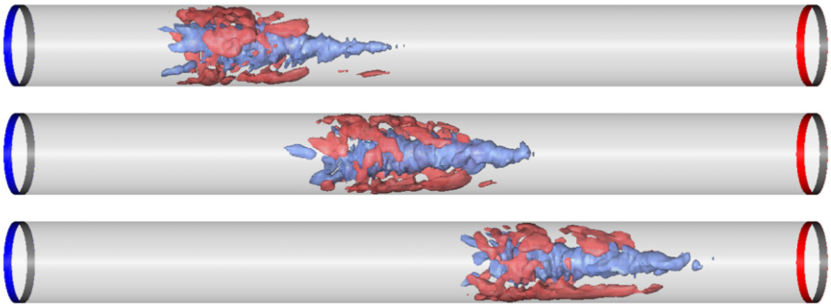

Figure 4. Three snapshots of a turbulent flow in a pipe at $Re = 1825$, moving from left to right. The snapshots are separated in time by $20D/u_{\text{cl}}$ with u_{cl} the centreline velocity. The red and blue regions indicate isosurfaces of downstream velocities somewhat faster or slower than the parabolic laminar profile. The aspect ratio is not shown to scale: the pipe is 50 diameters long.

or breaks up into two shorter ones which again grow to be $30D$ long (Hof 2004). Therefore, there is considerable robustness in the turbulent dynamics of spots:

Question 9. For Reynolds numbers below about 2700, the turbulence in a long pipe comes in the form of localized puffs. How is the dynamics of the puffs, their length selection and their boundary dynamics connected to the periodic coherent structures?

As a first step towards describing localized turbulence one can imagine some sort of Ginzburg–Landau type model for the dynamics of an envelope of the turbulent region, as in Prigent *et al* (2002), but this is unsatisfactory unless the equations and their coefficient can be derived from the underlying Navier–Stokes dynamics. It is interesting to note that a similar localization phenomenon can be studied in plane Couette flow (Barkley and Tuckerman 2005).

9. Final remarks

The discussion in the preceding section has focussed on pipe flow, but the occasional references to results for other flow show that there are several other related problems. One is plane Couette flow between parallel plates in relative motion, where the laminar profile is also linearly stable for all Reynolds numbers (Dauchot and Daviaud 1994, 1995, Kreiss *et al* 1994, Bottin *et al* 1998, Schmiegel and Eckhardt 1997, Dauchot and Vioujard 2000). Pressure driven flow between parallel plates, plane Poiseuille flow, has a parabolic profile and a curious linear instability at Reynolds numbers near 5772. However, turbulence is observed at Reynolds numbers of about 1000, and hence can follow a mechanism similar to the ones described here. For instance, there are travelling waves in these flows (Ehrenstein and Koch 1991, Waleffe 1998, 2001). The flow between rotating concentric cylinders, Taylor–Couette flow, is a closed flow geometry where a stable laminar profile and a turbulent dynamics coexist (Faisst and Eckhardt 2000, Hristova *et al* 2002). At present there seem to be many similarities and connections between these flows, suggesting that these flows follow the same route to turbulence.

The ultimate challenge is to understand the full hydrodynamic flow and hence the properties of the Navier–Stokes equation. But it is good to know that for the development and test of ideas there are models on different levels of complexity, from ordinary differential equations with varying numbers of degrees of freedom to simplified partial differential equations, see, e.g. Waleffe (1997), Brosa and Grossmann (1999), Eckhardt and Mersmann (1999), Manneville and Locher (2000), Moehlis *et al* (2004, 2005), Smith *et al* (2005) and Lagha and Manneville (2007).

Osborne Reynolds motivated his study not only with the obvious practical relevance of pipe flow but also with his interest in the nature of the transition. He could not have anticipated that while he was among the first to describe a transition in detail, his example would be among the last to be explained. But in hindsight it is clear that any serious explanation of the transition to turbulence in pipe flow requires quite a bit of nonlinear dynamics. Fortunately for dynamical systems theory, the transition in pipe flow is not only at the receiving end: the complexity of the edge state and the intriguing possibilites for the connections between the different states in the high-dimensional state space can be expected to stimulate nonlinear dynamics of high-dimensional systems as well.

Acknowledgments

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