The self-induced dynamics of vortex patches

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Abstract

The aim of this work is to propose and apply some analytical relations between the self-induced dynamics of a vortex patch and the Schwarz function of its frontier.

The first use of the Schwarz function in the kinematics of planar vortices dates back to the nineteen eighties [1]. By definition, the Schwarz function $\mathbf{S}(\mathbf{z})$ of an analytical closed curve ∂P bounding a simple connected domain P in a complex plane where $\mathbf{z} = \mathrm{x} + \mathbf{i}\ \mathrm{y}$ is obtained via the unique locally analytical continuation of the conjugate position $\bar{\mathbf{z}}$ of the points on ∂P which is possible on a strip-like annular region around the curve [2]. In the vortex dynamics framework, ∂P is assumed to coincide with the frontier of a uniform vortex P, which has unitary vorticity for the sake of simplicity.

In classical formulations, $\mathbf{S}(\mathbf{z})$ is broken up into the sum of two functions: $\mathbf{S} = \mathbf{F} + \mathbf{G}$, with analytical $\mathbf{F}$ inside and analytical $\mathbf{G}$ outside ∂P. The explicit formula for the conjugate self-induced velocity $\bar{\mathbf{u}}$ of P follows from such a decomposition:

$$\bar{\mathbf{u}}(\mathbf{z}) = \begin{cases} [\bar{\mathbf{z}} - \mathbf{F}(\mathbf{z})]/(\mathbf{2\,i}) & \text{inside P} \\ \mathbf{G}(\mathbf{z})/(\mathbf{2\,i}) & \text{outside P} \end{cases} \tag{1}$$

The continuity across the vortex boundary is fulfilled by the relation $\mathbf{G} = \bar{\mathbf{z}} - \mathbf{F}$, which is true on ∂P.

Form (1) relates the velocity and Schwarz function in an appealing way. However, it is often quite complicated to identify the $\mathbf{F}$ and $\mathbf{G}$ functions. We propose an alternative approach, in which $\bar{\mathbf{u}}$ and $\mathbf{S}$ are linked through a Cauchy integral and the splitting of $\mathbf{S}$ is avoided. We recover some results of previous literature [3] and frame them in a suitable mathematical picture. Furthermore, we show how the tools that are already available to solve the motion of patches in Euler flows (say, the Point vortex method, the Contour dynamics, the Elliptical model and the study using Schwarz functions) can be bundled together by introducing the concept of *balayage.*

The starting point is the contour dynamics form of the velocity induced by a uniform vortex P :

$$\mathbf{u}(\mathbf{z}) = -\int_{\partial \mathbf{P}} \mathbf{dz}'\ \mathbf{G}(\mathbf{z} - \mathbf{z}')\ , \tag{2}$$

where $G(\mathbf{z}) = \log|\mathbf{z}|/(\mathbf{2}\pi)$, $|\mathbf{z}|=(x^2+y^2)^{1/2}$ and $d\mathbf{z}'$ is the curve element. Here, the patch motion is defined by the time evolution of its boundary by starting from a smooth and finite-length boundary at the initial time ($t = 0$). If we consider a parameter σ_t on ∂P at time t, since the contour motion is material, σ_t can be written as $\sigma_t = \sigma_t(\sigma_0, t)$. The Lagrangian representation of the position on the vortex boundary:

$$\mathbf{z} = \mathbf{z}[\sigma_{\mathbf{t}}(\sigma_{\mathbf{0}}, \mathbf{t}), \mathbf{t}] = \mathbf{z}^{\wp}(\sigma_{\mathbf{0}}, \mathbf{t})$$

becomes natural and the velocity evaluated on that point, *i.e.* $\mathbf{u}(\mathbf{z}^{\wp})$, gives the Lagrangian velocity $\partial_t \mathbf{z}^{\wp}$. It follows that the motion of ∂P is the solution of the Cauchy problem:

$$\begin{cases} \partial_t \mathbf{z}^{\wp}(\sigma_{\mathbf{0}}, \mathbf{t}) = & \mathbf{u}[\mathbf{z}^{\wp}(\sigma_{\mathbf{0}}, \mathbf{t})] \\ \mathbf{z}^{\wp}(\sigma_{\mathbf{0}}, \mathbf{0}) & \text{given.} \end{cases} \tag{3}$$

By starting from the contour dynamics form of the velocity (2) and evaluating its tangent derivative [4,5], an integral relation between the Schwarz function and velocity is written in terms of the characteristic function χ_P of the domain P (which holds 1 inside P, 0 outside and 1/2 on the boundary) as:

$$\bar{\mathbf{u}}(\mathbf{z}, \bar{\mathbf{z}}) = \frac{\mathbf{1}}{\mathbf{2i}} \left[\chi_{\mathbf{P}}(\mathbf{z})\ \bar{\mathbf{z}} + \frac{\mathbf{1}}{\mathbf{2\pi i}} \int_{\partial \mathbf{P}} \mathbf{dz}'\ \frac{\mathbf{S}(\mathbf{z}')}{\mathbf{z} - \mathbf{z}'} \right]\ , \tag{4}$$

the integral becoming a Cauchy one when $\mathbf{z}$ lies on ∂P. Moreover, once the splitting of $\mathbf{S} = \mathbf{F} + \mathbf{G}$ is inserted into (4), the original formulation (1) is recovered. Extensive applications to sample families of vortices (see

Fig.1) are discussed in the paper, together with a study that introduces the concept of their corresponding *mother body.*

The use of the Schwarz function is also extended to the investigation of the vortex dynamics by introducing an evolution equation for $\mathbf{S}$:

$$\partial_t \mathbf{S} = \bar{\mathbf{u}} - \mathbf{u}\, \partial_{\mathbf{z}} \mathbf{S}\ . \tag{5}$$

The use of equation (5), together with the integral form of the velocity (4), opens the way to a re-reading of the dynamics of a uniform vortex in terms of the motion of the singular points of the corresponding Schwarz function. This is the subject of our present research.

As a sample case, let us consider an elliptical uniform vortex with semiaxes a and b and one focus in the point $\mathbf{c} = \mathbf{c}\exp(\mathbf{i}\theta)$, where θ is an angular coordinate. The quantities a, b and θ are assumed as functions of time. The Schwarz function is written as:

$$\mathbf{S} = \frac{\exp(-\mathbf{2i}\theta)}{\mathbf{c^2}} \left[(\mathbf{a^2}+\mathbf{b^2})\mathbf{z} - \mathbf{2ab}\sqrt{\mathbf{z^2}-\mathbf{c^2}}\right]\ . \tag{6}$$

Once the conjugate of the velocity has been calculated through equation (4), the right-hand-side of equation (5) becomes:

$$\bar{\mathbf{u}} - \mathbf{u}\, \partial_{\mathbf{z}} \mathbf{S} = \frac{\mathbf{2i}\exp(-\mathbf{2i}\theta)}{\mathbf{c^2}} \left[-(\mathbf{a^2}+\mathbf{b^2})\,\mathbf{z} + \mathbf{2ab}\,\sqrt{\mathbf{z^2}-\mathbf{c^2}} + \frac{\mathbf{abc^2}}{\sqrt{\mathbf{z^2}-\mathbf{c^2}}}\right] \frac{\mathbf{ab}}{(\mathbf{a}+\mathbf{b})^2}\ . \tag{7}$$

The time derivative of the Schwarz function (6) is instead given by:

$$\partial_t \mathbf{S} = \frac{\mathbf{2i}\exp(-\mathbf{2i}\theta)}{\mathbf{c^2}} \left[-(\mathbf{a^2}+\mathbf{b^2})\,\mathbf{z} + \mathbf{2ab}\,\sqrt{\mathbf{z^2}-\mathbf{c^2}} + \frac{\mathbf{abc^2}}{\sqrt{\mathbf{z^2}-\mathbf{c^2}}}\right] \dot{\theta} + \text{terms in } \dot{a} \text{ and } \dot{b}\ . \tag{8}$$

Comparing relations (7) and (8), it follows that a and b are constant in time and the angular velocity $\dot{\theta}$ assumes the classical form $ab/(a+b)^2$.

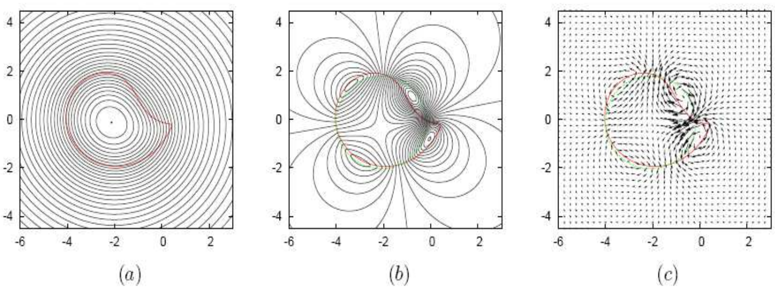

Figure 1: Sample result of flow field evaluable analytically. In a, in black, streamlines of a non trivial vortex (of red frontier). Differential streamlines ($\triangle\psi = 0.01$) and velocities (with a scale factor of 3) with respect to the equivalent Rankine vortex (of green frontier) are shown in b and c.

References

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